A Basis for 3 by 3 Symmetric Matrices

The real 3 by 3 matrices form a vector space M. The symmetric matrices in M form a subspace S. If you add two symmetric matrices, or multiply by real numbers, the result is still a symmetric matrix. **Problem: Find a basis for *S*.**

When I asked this question on an exam, I realized that a key point needs to be emphasized: **The basis "vectors" for *S* must lie in the subspace.** They are 3 by 3 symmetric matrices! Then there are two requirements:

1. The basis vectors must be linearly independent.
2. Their combinations must produce every vector (matrix) in S.

Here is one possible basis (all symmetric) for this example:

$$S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad S_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad S_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S_4 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad S_5 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad S_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Since this basis contains 6 vectors, the **dimension of *S* is 6.**

Question: Find a basis for the subspace AS of 3 by 3 antisymmetric matrices (with $A^{\mathrm{T}} = -A$). What is its dimension?

Bases for S and AS together give a basis for the whole space M (all 3 by 3 matrices). Write the upper triangular all-ones matrix U as a symmetric matrix plus an antisymmetric matrix.