Final Examination in Linear Algebra: 18.06

Dec 21, 2000   9:00 – 12:00   Professor Strang

Your name is: ____________________________

Grading
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Please circle your recitation:

1) M2  2-131 Holm   2-181  3-3665  tsh@math
2) M2  2-132 Dumitriu 2-333  3-7826  dumitriu@math
3) M3  2-131 Holm   2-181  3-3665  tsh@math
4) T10 2-132 Ardila  2-333  3-7826  fardila@math
5) T10 2-131 Czyz   2-342  3-7578  czyz@math
6) T11 2-131 Bauer  2-229  3-1589  bauer@math
7) T11 2-132 Ardila  2-333  3-7826  fardila@math
8) T12 2-132 Czyz   2-342  3-7578  czyz@math
9) T12 2-131 Bauer  2-229  3-1589  bauer@math
10) T1  2-132 Ingerman 2-372  3-4344  ingerman@math
11) T1  2-131 Nave   2-251  3-4097  nave@math
12) T2  2-132 Ingerman 2-372  3-4344  ingerman@math
13) T2  1-150 Nave   2-251  3-4097  nave@math

Answer all 8 questions on these pages (25 parts, 4 points each). This is a closed book exam. Calculators are not needed in any way and therefore not allowed (to be fair to all). Grades are known only to your recitation instructor. Best wishes for the holidays and thank you for taking 18.06. GS
(a) Explain why every eigenvector of $A$ is either in the column space $C(A)$ or the nullspace $N(A)$ (or explain why this is false).

(b) From $A = SAS^{-1}$ find the eigenvalue matrix and the eigenvector matrix for $A^T$. How are the eigenvalues of $A$ and $A^T$ related?

(c) Suppose $Ax = 0$ and $A^T y = 2y$. **Deduce that** $x$ **is orthogonal to** $y$. You may prove this directly or use the subspace ideas in (a) or the eigenvector matrices in (b). Write a clear answer.
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(a) Suppose $A$ is a symmetric matrix. If you first subtract 3 times row 1 from row 3, and after that you subtract 3 times column 1 from column 3, is the resulting matrix $B$ still symmetric? Yes or not necessarily, with a reason.
(b) Create a symmetric positive definite matrix (but not diagonal) with eigenvalues 1, 2, 4.
(c) Create a nonsymmetric matrix (if possible) with those eigenvalues. Create a rank-one matrix (if possible) with those eigenvalues.
Gram-Schmidt is $A = QR$ (start from rectangular $A$ with independent columns, produce $Q$ with orthonormal columns and upper triangular $R$). The problem is to produce the same $Q$ and $R$ from ordinary (symmetric) elimination on $A^T A$ which gives

$$A^T A = L D L^T = R^T R \quad \text{(with } R = \sqrt{D L^T}).$$

(a) How do you know that the pivots are positive, so $\sqrt{D}$ gives real numbers?

(b) From $A^T A = R^T R$ show that the matrix $Q = AR^{-1}$ has orthonormal columns (what is the test?). Then we have $A = QR$.

(c) Apply Gram-Schmidt to these vectors $a_1$ and $a_2$, producing $q_1$ and $q_2$. Write your result as $QR$:

$$a_1 = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}, \quad a_2 = \begin{bmatrix} \sin \theta \\ 0 \end{bmatrix}.$$
The Fibonacci numbers $F_0, F_1, F_2, F_3, F_4, \ldots$ are 0, 1, 1, 2, 3, \ldots and they obey the rule $F_{k+2} = F_{k+1} + F_k$. In matrix form this is

\[
\begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} \text{ or } u_{k+1} = Au_k.
\]

The eigenvalues of this particular matrix $A$ will be called $a$ and $b$.

(a) What quadratic equation connected with $A$ has the solutions (the roots) $a$ and $b$?

(b) Find a matrix that has the eigenvalues $a^2$ and $b^2$. What quadratic equation has the solutions $a^2$ and $b^2$?

(c) If you directly compute $A^4$ you get

\[
A^4 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}.
\]

Make a guess at the entries of $A^k$, involving Fibonacci numbers. Then multiply by $A$ to show why your guess is correct. What is the determinant of $A^k$ (not a hard question!)?
Suppose $A$ is 3 by 4 and its reduced row echelon form is $R$:

$$
R = \begin{bmatrix}
1 & 3 & 0 & 2 \\
0 & 0 & 1 & 5 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

(a) The four subspaces associated with the original $A$ are $N(A)$, $C(A)$, $N(A^T)$, and $C(A^T)$. Give the dimension of each subspace and if possible give a basis.

(b) Find the complete solution (when is there a solution?) to the equations

$$
\begin{bmatrix}
1 & 3 & 0 & 2 \\
0 & 0 & 1 & 5 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} =
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}.
$$

(c) Find a matrix $A$ with no zero entries (if possible) whose reduced row echelon form is this same $R$. 

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Suppose $A$ is a 3 by 3 matrix and you know the three outputs $y_1 = Ax_1$ and $y_2 = Ax_2$ and $y_3 = Ax_3$ from three independent input vectors $x_1, x_2, x_3$.

(a) Find the matrix $A$ using this hint: Put the vectors $x_1, x_2, x_3$ into the columns of a matrix $X$ and multiply $AX$. Why did I require the $x$’s to be independent?

(b) Under what condition on $A$ will the outputs $y_1, y_2, y_3$ be a basis for $R^3$? Explain your answer.

(c) If $x_1, x_2, x_3$ is the input basis and $y_1, y_2, y_3$ is the output basis, what is the matrix $M$ that represents this same linear transformation (defined by $T(x_1) = y_1$, $T(x_2) = y_2$, $T(x_3) = y_3$)?
7. (a) Find the eigenvalues of the antidiagonal matrix

\[
A = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

(b) Find as many eigenvectors as possible, with the best possible properties. Are there 4 independent eigenvectors? Are there 4 orthonormal eigenvectors?

(c) What is the rank of \(A + 2I\)? What is the determinant of \(A + 2I\)?
(a) If $U \Sigma V^T$ is the singular value decomposition of $A$ ($m$ by $n$) give a formula for the best least squares solution $\tilde{x}$ to $Ax = b$. (Simplify your formula as much as possible).

(b) Write down the equations for the straight line $b = C + Dt$ to go through all four of the points $(t_1, b_1)$, $(t_2, b_2)$, $(t_3, b_3)$, $(t_4, b_4)$. Those four points lie on a line provided the vector $b = (b_1, b_2, b_3, b_4)$ lies in \underline{subspace spanned by the columns of $A$}.

(c) Suppose $S$ is the subspace spanned by the columns of some $m$ by $n$ matrix $A$. Give the formula for the projection matrix $P$ that projects each vector in $\mathbb{R}^m$ onto the subspace $S$. \underline{Explain where this formula comes from and any condition on $A$ for it to be correct.}

(d) Suppose $x$ and $y$ are both in the row space of a matrix $A$, and $Ax = Ay$. Show that $x - y$ is in the nullspace of $A$. Then prove that $x = y$. 
