Math 18.06 Quiz 3 Solutions

1 (30 pts.) (a)

\[ A = SAS^{-1} = \begin{bmatrix} 1 & 6 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & -6 & -8 \\ & & & \\ & & & \end{bmatrix} \]

(b)

\[ A^\infty = \begin{bmatrix} 1 & 6 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & -6 & -8 \\ & & & \\ & & & \end{bmatrix} = \begin{bmatrix} 0 & 6 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

(c) The eigenvalues of \( B \) must both be 1. Suppose \( B \) has the Jordan form \( J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \), with \( B = MJM^{-1} \). Then \( B^n = MJ^nM^{-1} \) and \( J^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \), which cannot converge. So \( B \) can NOT have Jordan Form \( J \). The only alternative is that \( B \) has Jordan form \( I \), in which case \( B = MIM^{-1} = I \)
2 (40 pts.)  (a) \( S^{-1} = S^T \), so \( A = S \Lambda S^T \) is symmetric. Singular values are always nonnegative, so from \( \Lambda = \Sigma \) the eigenvalues of \( A \) are nonnegative, so \( A \) is symmetric positive semidefinite. It can be singular (the all zeros matrix is an example).

(b) The eigenvalues of a projection matrix are either 0 or 1, and their sum is 2, so they must be 1, 1, 0. For example

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1/2 & 1/2 \\
0 & 1/2 & 1/2
\end{bmatrix}
\]

(c) \( A^T A = \begin{bmatrix} 25 & 0 \\ 0 & 49 \end{bmatrix} \) so the singular values are 7 and 5. So

\[
V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

and

\[
A = \begin{bmatrix} 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

(d) 1. The eigenvalues of \( A \) are 1,1,2 - the same as the eigenvalues of \( B \).
2. \( A \) might or might not be diagonalizable
3. \( A \) might or might not be symmetric
4. \( A \) definitely (!) has positive eigenvalues. However it might not be symmetric, so \( A \) might or might not be positive definite.
3 (30 pts.) (a) The eigenvalues are $0, \sqrt{2}i, -\sqrt{2}i$. They are all pure imaginary (including zero!) because $A$ is skew symmetric.

(b) The general solution is $\vec{u}(T) = c_1 x_1 + c_2 e^{\sqrt{2}iT} x_2 + e^{-\sqrt{2}iT} x_3$

(c) $e^{i\theta} = \cos \theta + i \sin \theta$. This function has a period of $2\pi$, so when $\sqrt{2}T = 2n\pi$, we have $\vec{u}(T) = \vec{u}(0)$. In particular, $T$ can be $\sqrt{2}\pi$. 