Your name is:  ____________________________

Please circle your recitation:

1) M2  2-131  P.-O. Persson  2-088  2-1194  persson
2) M2  2-132  I. Pavlovsky  2-487  3-4083  igorvp
3) M3  2-131  I. Pavlovsky  2-487  3-4083  igorvp
4) T10  2-132  W. Luo  2-492  3-4093  luowei
5) T10  2-131  C. Boulet  2-333  3-7826  cilanne
6) T11  2-131  C. Boulet  2-333  3-7826  cilanne
7) T11  2-132  X. Wang  2-244  8-8164  xwang
8) T12  2-132  P. Clifford  2-489  3-4086  peter
9) T1  2-132  X. Wang  2-244  8-8164  xwang
10) T1  2-131  P. Clifford  2-489  3-4086  peter
11) T2  2-132  X. Wang  2-244  8-8164  xwang

The ten questions are worth 10 points each.

Thank you for taking 18.06!
The 4 by 6 matrix $A$ has all 2’s below the diagonal and elsewhere all 1’s:

$$A = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 1 & 1 & 1
\end{bmatrix}$$

(a) By elimination factor $A$ into $L$ (4 by 4) times $U$ (4 by 6).

(b) Find the rank of $A$ and a basis for its nullspace (the special solutions would be good).
Suppose you know that the 3 by 4 matrix $A$ has the vector $s = (2, 3, 1, 0)$ as a basis for its nullspace.

(a) What is the rank of $A$ and the complete solution to $Ax = 0$?

(b) What is the exact row reduced echelon form $R$ of $A$?
The following matrix is a projection matrix:

\[ P = \frac{1}{21} \begin{bmatrix} 1 & 2 & -4 \\ 2 & 4 & -8 \\ -4 & -8 & 16 \end{bmatrix}. \]

(a) What subspace does \( P \) project onto?
(b) What is the distance from that subspace to \( b = (1, 1, 1) \)?
(c) What are the three eigenvalues of \( P \)? Is \( P \) diagonalizable?
(a) Suppose the product of $A$ and $B$ is the zero matrix: $AB = 0$. Then the (1) space of $A$ contains the (2) space of $B$. Also the (3) space of $B$ contains the (4) space of $A$. Those blank words are

(1) ____________ (2) ____________ (3) ____________ (4) ____________

(b) Suppose that matrix $A$ is 5 by 7 with rank $r$, and $B$ is 7 by 9 of rank $s$. What are the dimensions of spaces (1) and (2)? From the fact that space (1) contains space (2), what do you learn about $r + s$?
Suppose the 4 by 2 matrix $Q$ has orthonormal columns.

(a) Find the least squares solution $\hat{x}$ to $Qx = b$.

(b) Explain why $QQ^T$ is not positive definite.

(c) What are the (nonzero) singular values of $Q$, and why?
Let $S$ be the subspace of $\mathbb{R}^3$ spanned by \[
\begin{bmatrix}
1 \\
2 \\
2
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
5 \\
4 \\
-2
\end{bmatrix}.
\]

(a) Find an orthonormal basis $q_1, q_2$ for $S$ by Gram-Schmidt.

(b) Write down the 3 by 3 matrix $P$ which projects vectors perpendicularly onto $S$.

(c) Show how the properties of $P$ (what are they?) lead to the conclusion that $Pb$ is orthogonal to $b - Pb$. 


(a) If \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) form a basis for \( \mathbb{R}^3 \) then the matrix with those three columns is _____.

(b) If \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \) span \( \mathbb{R}^3 \), give all possible ranks for the matrix with those four columns. __________.

(c) If \( \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3 \) form an orthonormal basis for \( \mathbb{R}^3 \), and \( T \) is the transformation that projects every vector \( \mathbf{v} \) onto the plane of \( \mathbf{q}_1 \) and \( \mathbf{q}_2 \), what is the matrix for \( T \) in this basis? Explain.
Suppose the $n$ by $n$ matrix $A_n$ has 3’s along its main diagonal and 2’s along the diagonal below and the $(1, n)$ position:

\[
A_4 = \begin{bmatrix}
3 & 0 & 0 & 2 \\
2 & 3 & 0 & 0 \\
0 & 2 & 3 & 0 \\
0 & 0 & 2 & 3
\end{bmatrix}.
\]

Find by cofactors of row 1 or otherwise the determinant of $A_4$ and then the determinant of $A_n$ for $n > 4$. 


There are six 3 by 3 permutation matrices $P$.

(a) What numbers can be the determinant of $P$? What numbers can be pivots?

(b) What numbers can be the trace of $P$? What four numbers can be eigenvalues of $P$?
Suppose $A$ is a 4 by 4 upper triangular matrix with 1, 2, 3, 4 on its main diagonal. (You could put all 1’s above the diagonal.)

(a) For $A - 3I$, which columns have pivots? Which components of the eigenvector $x_3$ (the special solution in the nullspace) are definitely zero?

(b) Using part (a), show that the eigenvector matrix $S$ is also upper triangular.