Course 18.06, Fall 2002: Quiz 3, Solutions

1. (a) One eigenvalue of $A = \text{ones}(5)$ is $\lambda_1 = 5$, corresponding to the eigenvector $x_1 = (1, 1, 1, 1, 1)$.
   Since the rank of $A$ is 1, all the other eigenvalues $\lambda_2, \ldots, \lambda_5$ are zero. Check: The trace of $A$ is 5.

(b) The initial condition $u(0)$ can be written as a sum of the two eigenvectors $x_1 = (1, 1, 1, 1, 1)$ and $x_2 = (-1, 0, 0, 0, 1)$, corresponding to the eigenvalues $\lambda_1 = 5$ and $\lambda_2 = 0$:
   
   $$u(0) = (0, 1, 1, 1, 2) = (1, 1, 1, 1, 1) + (-1, 0, 0, 0, 1) = x_1 + x_2.$$

   The solution to $\frac{du}{dt} = Au$ is then
   
   $$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 = (1, 1, 1, 1, 1) e^{5t} + (-1, 0, 0, 0, 1).$$

(c) The eigenvectors of $B = A - I$ are the same as for $A$, and the eigenvalues are smaller by 1:

   $$B x = (A - I) x = Ax - x = \lambda x - x = (\lambda - 1) x,$$

   where $x, \lambda$ are an eigenvector and an eigenvalue of $A$. The eigenvalues of $B$ are then $4, -1, -1, -1, -1$, the trace is $\sum \lambda_i = 0$, and the determinant is $\prod \lambda_i = 4$.

2. (a) $B$ is similar to $A$ when $B = M^{-1} AM$, with $M$ invertible. The exponential of $A$ is

   $$e^A = I + A + \frac{1}{2} A^2 + \frac{1}{6} A^3 + \cdots.$$

   Every power $B^k$ of $B$ is similar to the same power $A^k$ of $A$:

   $$B^k = M^{-1} AM M^{-1} AM \cdots M^{-1} AM = M^{-1} A^k M.$$

   Then

   $$e^B = I + B + \frac{1}{2} B^2 + \cdots = M^{-1} \left( I + A + \frac{1}{2} A^2 + \cdots \right) M = M^{-1} e^A M.$$

   It is also OK to show this using $e^A = S e^\Lambda S^{-1}$, although that assumes that the matrices are diagonalizable.

(b) The exponential of $A$ is

   $$e^A = S e^\Lambda S^{-1} = S \begin{bmatrix} e^0 & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^4 \end{bmatrix} S^{-1}.$$

   But this is an eigenvalue decomposition of $e^A$, so the eigenvalues are $1, e^2, e^4$.

   More generally, the eigenvalues of $e^A$ are the exponentials of the eigenvalues of $A$, and

   $$\det(e^A) = e^{\lambda_1} e^{\lambda_2} \cdots e^{\lambda_n} = e^{\lambda_1 + \lambda_2 + \cdots + \lambda_n} = e^{\text{tr}(A)}.$$
(a) For $A$ to be symmetric, $U$ has to be equal to $V$ (notice $V^T$ in the matrices):

$$\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta 
\end{bmatrix} = \begin{bmatrix}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha 
\end{bmatrix}.$$ 

Together with the restrictions on $\theta, \alpha$ this requires that $\theta = \alpha$. $A$ is then a positive definite symmetric matrix, since it is similar to $\begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$.

(b) The eigenvalues of $A^T A$ are the square of the singular values, that is, 81 and 16. The eigenvectors of $A^T A$ are the columns of $V$, that is, $(\cos \alpha, \sin \alpha)$ and $(-\sin \alpha, \cos \alpha)$. This can also be shown by multiplying $A^T A = V \Sigma^2 V^T$ and identifying this as the eigenvalue decomposition of $A^T A$.

4 (a) $A$ is singular, so one eigenvalue is 0. It is also a Markov matrix, so another eigenvalue is 1 (Motivation: Each column of $A$ sums to 1, so each column of $A - I$ sums to 0. $A - I$ then has an eigenvalue 0, and $A$ has an eigenvalue 1). The last eigenvalue is 0.5 since $\text{trace}(A) = \sum_i \lambda_i = 1.5$.

The eigenvectors are found by solving the following systems:

$$\lambda_1 = 1: \quad (A - \lambda_1 I)x_1 = \begin{bmatrix} -0.5 & 0.5 & 0.5 \\ 0.25 & -0.5 & 0 \\ 0.25 & 0 & -0.5 \end{bmatrix} x_1 = 0 \implies x_1 = (2,1,1),$$

$$\lambda_2 = 0.5: \quad (A - \lambda_2 I)x_2 = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.25 & 0 & 0 \\ 0.25 & 0 & 0 \end{bmatrix} x_2 = 0 \implies x_2 = (0,1,-1),$$

$$\lambda_3 = 0: \quad (A - \lambda_3 I)x_3 = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.25 & 0.5 & 0 \\ 0.25 & 0 & 0.5 \end{bmatrix} x_3 = 0 \implies x_3 = (2,-1,-1).$$

(b) Write the initial value as a linear combination of the eigenvectors:

$$u_0 = (6,0,6) = 3x_1 - 3x_2.$$ 

The distribution after $k$ steps is then

$$u_k = A^k u_0 = 3\lambda_1^k x_1 - 3\lambda_2^k x_2 = 3x_1 - 3 \cdot 0.5^k x_2 \to 3x_1 = (6,3,3) \text{ as } k \to \infty.$$