Questions from 18.06 Final, Fall 2003

1. Suppose $A = LU$ where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 5 & 0 & 5 & 1 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) What are the dimensions of the 4 fundamental subspaces associated with $A$?

(b) Give a basis for each of the 4 fundamental subspaces.
\[ N(A) \]

\[ R(A) \]

\[ C(A) \]

\[ N(A^T) \]
2. Let $F$ be the subspace of $\mathbb{R}^4$ given by

$$F = \{(x, y, z, w) : x - y + 2z + 3w = 0\}.$$ 

Let $P$ be the projection matrix for projecting onto $F$. (Many of the subquestions can be answered independently of the others.)

(a) Give an orthonormal basis $\{v_1, \cdots, v_k\}$ for the orthogonal complement to $F$.

(b) Find an orthonormal basis $\{w_1, \cdots, w_l\}$ for $F$. Explain how you proceed.
The following questions refer to the projection matrix $P$ for projecting onto $F$.

(c) What are the eigenvalues of $P$? Give them with their multiplicities.

(d) What is the projection of \[
\begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix}
\] onto $F$?
3. (a) Write down the $2 \times 2$ rotation matrix, $R(\theta)$, that rotates $\mathbb{R}^2$ in the counterclockwise direction by an angle $\theta$ (this matrix is a function of $\theta$).

(b) Compute the eigenvalues of $R(\theta)$. For which value(s) of $\theta$ are the eigenvalues real?

(c) What are the eigenvectors of $R(\theta)$.
(d) Write down the singular value decomposition of $R(\theta)$. 
4. (a) Give two $3 \times 3$ matrices $A$ and $B$ such that $AB$ is not equal to $BA$.

(b) Suppose $A$ and $B$ are $n \times n$ matrices with the same set of linearly independent eigenvectors $v_1, v_2, \cdots, v_n$. However, the eigenvalues might be different: $v_i$ is the eigenvector for the eigenvalue $\lambda_i$ of $A$ and the eigenvector for the eigenvalue $\mu_i$ of $B$. Show that $AB = BA$. 
5. Consider the differential equation \[
\begin{bmatrix}
\frac{du}{dt} \\
\frac{dv}{dt}
\end{bmatrix} = \begin{bmatrix}
0 & 3 \\
2 & -1
\end{bmatrix} \begin{bmatrix}
u \\
v
\end{bmatrix}.
\]

(a) Solve the differential equation and express \(u(t), v(t)\) as functions of \(u(0)\) and \(v(0)\).
(b) Find a linear transformation \[ \begin{bmatrix} p \\ q \end{bmatrix} = T \begin{bmatrix} u \\ v \end{bmatrix} \] such that the differential equation simplifies into two independent differential equations in p and in q (one relating \( \frac{dp}{dt} \) and p, the other relating \( \frac{dq}{dt} \) and q).

(c) Are there initial conditions \( u(0), v(0) \) that would make \( u(t) \) blow up? If yes, give one such value for \( u(0) \) and \( v(0) \).

(d) Are there initial conditions \( u(0), v(0) \) that would make \( u(t) \) go to 0? If yes, give one such value for \( u(0) \) and \( u'(0) \).

(a) The product of the pivots when performing Gauss-Jordan is equal to the determinant if we do not have to permute rows.
   True or False.

(b) Let $A$ be an $m \times n$ matrix whose columns are independent. Then $AA^T$ is positive definite.
   True or False.

(c) If $A$ is a symmetric matrix then the singular values are the absolute values of the nonzero eigenvalues.
   True or False.

(d) There exists a $5 \times 5$ unitary matrix with eigenvalues $1, 1 + i, 1 - i, i$ and $-i$.
   True or False.

(e) Suppose $V$ and $W$ are two vector spaces of dimension $n$. If $T$ is a linear transformation from $V$ to $W$ with only the 0 vector in the kernel, then for any basis of $V$, there exists an orthonormal basis of $W$ such that the resulting matrix representing $T$ is upper triangular.
   True or False.
7. Consider the following matrix $A$:

$$A = \begin{bmatrix} 0.5 & 0.4 & 0.2 \\ 0.4 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.6 \end{bmatrix}.$$ 

(c) Can you immediately tell one of the eigenvalues of $A$ (without computing them)? Explain.

(d) Compute the determinant of $A$.

(e) Find the eigenvalues of $A$ and the corresponding eigenvectors. (Check your answer.)
(f) Two out of the 3 eigenvectors of $A$ should be orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. How could you have explained this before computing the eigenvectors?

(g) Write an exact expression for $A^{100}$. 
8. Let

\[ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}. \]

For each of the following matrices, either complete it (find values for the non-diagonal elements) so that it becomes similar to \( A \), or explain why it is impossible to complete it to a matrix similar to \( A \). Circle whether you are able to complete it or not to a matrix similar to \( A \).

(a)

\[ B = \begin{bmatrix} 2 & . & . \\ . & 2 & . \\ . & . & 4 \end{bmatrix} \]

Able to complete it to similar?: Yes No

If yes, give a completion. If not, why not?
(b)

\[
C = \begin{bmatrix}
3 & \cdot & \cdot \\
\cdot & 3 & \cdot \\
\cdot & \cdot & 3
\end{bmatrix}.
\]

Able to complete it to similar?: Yes No
If yes, give a completion. If not, why not?