18.06, Fall 2003, Problem Set 4 Solutions

1. (a) False. counterexample: 
\[
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

(b) True. multiplication by scalar does not change the four fundamental spaces of a matrix.

(c) False. 
\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
1 & 0 \\
0 & 2
\end{bmatrix}
\]
are not multiples of one another but have the same fundamental subspaces.

2. (a) \(A'\) has an extra row \((0, -1, 0, 0, 1, 0)\) than \(A\).

(b) No, they do not differ. \(\text{Null}(A)\) is the space of constant vectors \((c, c, c, c, c)\), so is the null space of \(A'\).

(c) The dimension of the left null space will increase from 2 to 3 when adding \(c_8\). \(A^T y = 0\) is solved by loop currents. There are many different bases of the left nullspaces of \(A\) or \(A'\). To get a basis of the left null space of \(A\), we can take \(7 - 6 + 1 = 2\) small loops, e.g. \((0, 0, 1, 0, 1, 0, -1)\), and \((1, 1, 0, 0, 1, 1, 1)\). To get a basis of the left null space of \(A'\), we can take \(8 - 6 + 1 = 3\) small loops: \((0,0,1,0,0,0,-1,0),(0,1,0,0,0,0,1,0),(1,0,0,0,0,1,0,1)\).

3. (a) The column space has dim 3, the left null space has dim \(5 - 3 = 2\).

(b) We know that a basis the row space of \(A\) will consist of 3 linearly independent vectors, each orthogonal to the given 3 vectors forming a basis of the nullspace. So we could just guess 3 such linearly independent vectors. Or we could place the given vectors forming the basis of \(N(A)\) as the 3 rows of a \(3 \times 6\) matrix \(B\) and compute a basis of its nullspace \(N(B) = R(A)\) by first row reducing it to its reduced row echelon form. After elimination, we get the reduced row echelon form:

\[
\begin{bmatrix}
1 & 0 & 0 & -2 & 1 & 1 \\
0 & 1 & 0 & 1 & -1 & -1 \\
0 & 0 & 1 & 2 & -2 & -1
\end{bmatrix}
\]

Therefore one basis of \(N(B) = R(A)\) is \((2, -1, -2, 1, 0, 0), (-1, 1, 2, 0, 1, 0), (-1, 1, 1, 0, 0, 1)\). There are many others.

4. It suffices to find the null space of the matrix

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8
\end{bmatrix}
\]

Its reduced row echelon form is

\[
\begin{bmatrix}
1 & 0 & -1 & 2 \\
0 & 1 & 2 & 3
\end{bmatrix}
\]

So a basis of the null space is

\[
\begin{pmatrix}
1 \\
2 \\
-2 \\
1 \\
0
\end{pmatrix}, \begin{pmatrix}
2 \\
-3 \\
0 \\
1
\end{pmatrix}
\]

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This is a basis of $W^\perp$.

5. Let $A = [v_1, v_2, v_3]$. Then the projection matrix $P = A(A^T A)^{-1}A^T$ is

$$P = \frac{1}{4} \begin{bmatrix}
3 & 1 & -1 & 1 \\
1 & 3 & 1 & -1 \\
-1 & 1 & 3 & 1 \\
1 & -1 & 1 & 3
\end{bmatrix}.$$ 

Hence, the projection of $v$ onto the subspace spanned by $v_1, v_2$ and $v_3$ is $Pv = \frac{1}{4} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}$.

6. (a) The $n$ standard vectors $e_i$ for $i = 1, 2, \ldots, n$, where $e_i$ has 1 in the $i$-th entry and zero everywhere else.

(b) Matlab code:

```matlab
for I=1:50,
    A=randn(50,30);
    P=A*inv(A'*A)*A';
    %B=eye(50);
    W=P;
    rank(W);
    Q=W'*W;
    d=diag(Q);
    e=sqrt(d*ones(1,50)+ones(50,1)*d'-2*Q);
    ma=max(max(e));
    mi=min(min(e+ma*eye(50)));
    C(I)=ma/mi;
end
>> min(C)

ans =

1.5903
```

By repeating many times, one could get below 1.5 but any answer below 1.75 is valid.

7. $e$ is orthogonal to $p$. $||e||^2 = e^T (b-p) = e^T b = b^T b - p^T b$, simply using the fact that $e = b - p$. 
