18.06. Fall 2003. Quiz 2 Solutions.

1. (20 points) Suppose that

\[ A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 6 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \]

(a) (8 points) Find the dimensions of the four fundamental subspaces of \( A \).

**Solution:** \( A \) is \( 3 \times 4 \), so \( m = 3 \) and \( n = 4 \). As \( L \) is invertible, \( \text{rank}(A) = \text{rank}(U) = 2 \) (2 pivots). Thus, \( \text{dim}(C(A)) = 2 \), \( \text{dim}(R(A)) = 2 \), \( \text{dim}(N(A)) = 4 - \text{rank}(A) = 2 \) and \( \text{dim}(N(A^T)) = 3 - \text{rank}(A) = 1 \).

(b) (6 points) Find a basis for the row space \( R(A) \).

**Solution:** \( R(A) = R(U) \) and thus the first two rows of \( U \) form a basis of \( R(A) \): \( (1, 0, 2, 4) \) and \( (0, 1, -1, 1) \). Any other basis (2 vectors spanning the same subspace) is perfect too.

(c) (6 points) Find a basis for the column space \( C(A) \).

**Solution:** Since \( C(U) \) has \((1, 0, 0)\) and \( (0, 1, 0) \) as basis, the first two columns of \( L \) will be a basis of \( C(A) \): \((1, 1, 6)\) and \((0, 1, 3)\). Any other basis (2 vectors spanning the same subspace) is perfect too.

2. (15 points) Let \( a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, a_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \) and \( a_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \). Let \( V \) be the subspace of \( R^4 \) spanned by \( a_1, a_2, \) and \( a_3 \). Using Gram-Schmidt, find an orthonormal basis \( q_1, q_2, \) and \( q_3 \) of \( V \). Show your work.

**Solution:** First we get 3 orthogonal vectors \( u_1, u_2, \) and \( u_3 \), and then we normalize them to get \( q_1, q_2, \) and \( q_3 \). We have

\[ u_1 = a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \]

\[ u_2 = a_2 - \frac{a_2 \cdot u_1}{u_1 \cdot u_1} u_1, \]

\[ u_3 = a_3 - \frac{a_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{a_3 \cdot u_2}{u_2 \cdot u_2} u_2, \]

and

\[ q_1 = \frac{u_1}{\|u_1\|}, q_2 = \frac{u_2}{\|u_2\|}, q_3 = \frac{u_3}{\|u_3\|}. \]
\[ u_2 = a_2 - \frac{u_2^T a_2}{u_2^T u_1} u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \]

\[ u_3 = a_3 - \frac{u_2^T a_3}{u_1^T u_1} u_1 - \frac{u_2^T a_3}{u_2^T u_2} u_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 3 \\ 3 \end{bmatrix}. \]

After normalization we get:

\[ q_1 = \frac{u_1}{\|u_1\|} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \]

\[ q_2 = \frac{u_2}{\|u_2\|} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \]

\[ q_3 = \frac{u_3}{\|u_3\|} = \begin{bmatrix} -1/\sqrt{20} \\ 3/\sqrt{20} \\ 1/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix}. \]

3. (35 points) Let \( a \) be a (column) vector in \( \mathbb{R}^3 \) and \( \ell \) the line consisting of all multiples of \( a \). Let \( P \) be the \( 3 \times 3 \) projection matrix for projecting \( \mathbb{R}^3 \) onto the line \( \ell \). In these subquestions, you do not need to justify your answer.

(a) (3 points) Write an expression for \( P \) in terms of \( a \).

**Solution:** \( P = \frac{1}{a^T a} a a^T \) or \( P = a(a^T a)^{-1} a^T \)

(b) (4 points) Write an expression for the distance from a point \( b \) to the line \( \ell \).

**Solution:** \( \|b - P b\| \) or \( \|b - \frac{a^T b}{a^T a} a\| \)

(c) (3 points) Give a basis for the column space \( C(P) \). What is the dimension of \( C(P) \)?

**Solution:** The vector \( a \) forms a basis, and \( C(P) \) has dimension 1.
(d) (4 points) Describe geometrically what the nullspace $N(P)$ of $P$ is. What is its dimension?

**Solution:** $N(P)$ is the orthogonal complement to the line $\ell$; it is the plane orthogonal to $\ell$. Its dimension is 2.

(e) (3 points) What is the rank of $P$?

**Solution:** 1 (since $C(P)$ has dimension 1).

(f) (3 points) Is $P$ invertible?

**Solution:** No (since rank is not 3).

(g) (3 points) What is the projection matrix $Q$ for projecting onto the orthogonal complement to $\ell$? Write an expression in terms of $P$.

**Solution:** $Q = I - P$.

(h) (4 points) What is rank of $PQ$?

**Solution:** 0 (since $PQ = P(I - P) = P - P^2 = P - P = 0$).

(i) (4 points) What are the eigenvalues of $P$? Give an eigenvector corresponding to the largest eigenvalue.

**Solution:** 0, 0 and 1. $a$ is an eigenvector corresponding to $\lambda = 1$.

(j) (4 points) What is $\det(P + 2I)$?

**Solution:** 12 (since the eigenvalues of $P$ differ from those of $P$ by 2 units, thus they are 2, 2 and 3, and the determinant is the product of the eigenvalues).

4. (15 points) Compute the determinant of

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 \\ 0 & 3 & 3 & 3 \\ 0 & 0 & 4 & 4 \end{bmatrix}.$$  

Show your work.

**Solution:** Let’s divide row $i$ by $i$. Thus $\det(A)$ is 24 times the determinant of

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$
Let’s do elimination. First subtract row 1 from row 2 to get:

\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
\end{bmatrix}
\]

Now the only permutation in the big formula that will be non-zero has column numbers \((1,3,2,4)\) (which is odd), so \(\det(B) = -1\) and \(\det(A) = -24\).

5. (10 points) Let

\[
A = \begin{bmatrix}
0 & -1 & -1 \\
a & b & c \\
-1 & -1 & 2 \\
\end{bmatrix},
\]

where \(a, b\) and \(c\) are not given. We are told that (i) 2 is an eigenvalue of \(A\), (ii) \(\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\) is an eigenvector and (iii) \(\det(A) = -6\).

(a) (6 points) What are all eigenvalues of \(A\)?

**Solution:** Since \(v = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\) is an eigenvector of \(A\), its corresponding eigenvalue \(\lambda\) satisfies \(Av = \begin{bmatrix} -2 \\ 2a + b + c \\ -1 \end{bmatrix} = \lambda v\), and thus \(\lambda = -1\).

The determinant is the product of the eigenvalues, and therefore the third eigenvalue is 3. The three eigenvalues are thus \(-1, 2\) and 3.

(b) (4 points) What are the values of \(a, b\) and \(c\)?

**Solution:** The trace of \(A\) is \(2 + b\) and is equal to the sum of the eigenvalues (\(= 4\)), so \(b = 2\). From \(Av = -v\), we get \(2a + b + c = -1\), or \(2a + c = -3\). \(\det(A) = 2a + b - (-a + 2) = 3a + c - 2 = -6\) or \(3a + c = -4\). This implies that \(a = c = -1\).