**18.06, Fall 2004, Problem Set 10 Solutions**

1. (13 pts.)

   Consider the differential equation $\frac{du}{dt} = Au$ where $u$ is 2-dimensional and

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

   (a) The characteristic polynomial of $A$ is

$$\det \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 + 1,$$

   and therefore the eigenvalues are $\lambda_1 = i$ and $\lambda_2 = -i$. The eigenvectors are $v_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$ (or any complex multiple of it) and $v_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$.

   (b) Since $A$ has distinct eigenvalues, it is diagonalizable and we have that $A = V\Lambda V^{-1}$ where

$$V = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix},$$

   and

$$\Lambda = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

   The inverse of $V$ is:

$$V^{-1} = \frac{1}{2} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix}.$$

   (Observe that since the columns of $V$ are orthogonal but not of unit norm, $V^{-1}$ is almost $V^H$; we could have scaled $V$ by $\frac{1}{\sqrt{2}}$ to make its inverse equal to its Hermitian.)

   Now, as $A$ is diagonalizable, we can compute $e^{At}$ by $Ve^{\Lambda t}V^{-1}$:

$$\begin{aligned} e^{At} &= \frac{1}{2} \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} -ie^{it} & ie^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{it} + e^{-it} & -ie^{it} + ie^{-it} \\ ie^{it} - ie^{-it} & e^{it} + e^{-it} \end{bmatrix} \\ &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & cos(t) \end{bmatrix}. \end{aligned}$$

(c) Assuming $u(0) = \begin{bmatrix} 9 \\ 2 \end{bmatrix}$ we get

$$u(t) = e^{At}u(0) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & cos(t) \end{bmatrix} \begin{bmatrix} 9 \\ 2 \end{bmatrix} = \begin{bmatrix} 9\cos(t) + 2\sin(t) \\ -9\sin(t) + 2\cos(t) \end{bmatrix}.$$

(d) The differential equation is not stable (since the eigenvalues have their real part equal to 0, and not strictly less than 0).

(e) It will be a circle, as $u_1^2(t) + u_2^2(t) = 9^2 + 2^2 = 40$.

2. (10 pts.) Let

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Since $A$ is upper triangular, the eigenvalues are just the diagonal elements (as $\det(A - \lambda I)$ is the product of the diagonal elements of $A - \lambda I$). Thus all eigenvalues are 0.

(b) The nullspace of $A$ has dimension 1, so we have only 1 linearly independent eigenvector.

(c) To compute $e^{tA}$, we have to use the infinite series $e^{At} = I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{6}A^3 + \cdots$. The powers of $A$ are:

$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and all powers $A^k$ with $k \geq 4$ are equal to 0. Thus,

$$\begin{aligned} e^{tA} &= I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{6}A^3 \\ &= I + t\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{t^2}{2}\begin{bmatrix} 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{t^3}{6}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & t & 2t + \frac{t^2}{2} & 3t + 2t^2 + \frac{1}{6}t^3 \\ 0 & 1 & t & 2t + \frac{t^2}{2} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

3. (13 pts.) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}.$$

(a) To compute an $LDL^T$ factorization of $A$, we need to do eliminations to transform $A$ into an upper triangular matrix. Pivoting on $(1,1)$, we get:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & -5 & -7 \end{bmatrix}.$$

Pivoting on $(2,2)$, we get:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 18 \end{bmatrix}.$$

This is equal to $DU$, where

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 18 \end{bmatrix}.$$

Now, since $A$ is symmetric, we know that the $LDU$ factorization will be $LDL^T$, and thus we have

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 5 & 1 \end{bmatrix}.$$

(b) As all the row sums are 6, one of the eigenvalue is 6 (say $\lambda_1$), with corresponding eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. After scaling we get that the eigenvector is:

$$v_1 = \begin{bmatrix} 1\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.$$

To get the others, we could compute the characteristic polynomial. But let's try to compute them otherwise. The trace of $A$ is 6, so the sum of the other 2 eigenvalues is 0, and we have $\lambda_2 = -\lambda_3$. The product of the eigenvalues is the determinant of $A$, which equals to $-18$ (either we could compute it explicitly or remember that it is the product of the pivots after elimination (modulus a possible sign change if we have done permutations)). Thus, $\lambda_2\lambda_3 = -3$ and we get $\lambda_2 = \sqrt{3}$ and $\lambda_3 = -\sqrt{3}$.

Let's now get an eigenvector for $\lambda_2$. We have:

$$A - \lambda_2 I = \begin{bmatrix} 1-\sqrt{3} & 2 & 3 \\ 2 & 3-\sqrt{3} & 1 \\ 3 & 1 & 2-\sqrt{3} \end{bmatrix}.$$

We need to find the nullspace, so we perform eliminations. Adding the first row times$(1+\sqrt{3})$ to the second, we get:

$$\begin{bmatrix} 1-\sqrt{3} & 2 & 3 \\ 0 & 5+\sqrt{3} & 4+3\sqrt{3} \\ * & * & * \end{bmatrix}.$$

We don't care about the 3rd row, as it will become 0 after further eliminations. The special solution here is

$$\begin{bmatrix} \frac{\sqrt{3}-1}{2} \\ -\frac{1+\sqrt{3}}{2} \\ 1 \end{bmatrix}.$$

We can scale it by $\sqrt{3}$ to get an eigenvector of unit norm:

$$v_2 = \begin{bmatrix} \frac{1}{2} - \frac{1}{2\sqrt{3}} \\ -\frac{1}{2\sqrt{3}} - \frac{1}{2} \\ \frac{1}{\sqrt{3}} \end{bmatrix}.$$

We do it similarly for $\lambda_3$. In fact, we can easily guess $v_3$ now as it is orthogonal to both $v_1$ and $v_2$ (since $A$ is symmetric). We get:

$$v_3 = \begin{bmatrix} -\frac{1}{2\sqrt{3}} - \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}.$$

(c) We just have to take for $Q$ the matrix whose columns are $v_1, v_2, v_3$ and for $\Lambda$ the diagonal matrix of eigenvalues.

$$\Lambda = \begin{bmatrix} 6 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & -\sqrt{3} \end{bmatrix},$$

and

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{2} - \frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} - \frac{1}{2} \\ \frac{1}{\sqrt{3}} & -\frac{1}{2\sqrt{3}} - \frac{1}{2} & \frac{1}{2} - \frac{1}{2\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

(d) In both cases we get -18, and this is the determinant of $A$; the determinant of $A$ is both the product of teh pivots and of the eigenvalues.

4. (4 pts.) Since $A$ is symmetric then $A = tI$ is also symmetric. Furthermore, for any eigenvalue $\lambda_i$ of $A$, we have that $\lambda_i + t$ is an eigenvalue of $A + tI$ and vice versa (since $(A+tI)v = \lambda v + tv$ for $v$ an eigenvalue of $A$ corresponding to $\lambda$). So, if $t > -\min_i \lambda_i$ then $A + tI$ has all its (real) eigenvalues greater than 0, and thus is positive definite.