18.06, Fall 2004, Problem Set 2 Solutions

1. (8 pts.)

   (a) Their inverses are:

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{bmatrix},$$

$$E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}.$$

   (b) M^{-1} has the multipliers below the diagonal:

$$M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 7 & -3 & 1 \end{bmatrix}.$$

2. (8 pts.)

   (a) The inverse of A^2 is $(A^{-1})^2 = A^{-1}A^{-1}$ since $A^2(A^{-1}A^{-1}) = A(AA^{-1})A^{-1} = AA^{-1} = I$.

   (b) No, A is always invertible if A^2 is invertible. Indeed, the inverse of A is AB where $B = (A^2)^{-1}$ since $A(AB) = A^2B = I$. (Notice that the same argument shows that the inverse of A is also BA and thus here A and B commute: $AB = BA$.)

   Another way of justifying that A has an inverse is (by contradiction) to say that if it had no inverse then there would be an $x \neq 0$ with $Ax = 0$ (see lecture on inverses). This would imply that $A^2x = A(Ax) = A0 = 0$ and thus we have a nonzero x with $A^2x = 0$. This would mean that A^2 is singular, a contradiction.

3. (7 pts.) Using $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, we get $E_{21}A = \begin{bmatrix} 3 & 2 & 6 \\ 0 & 0 & -1 \\ 3 & 3 & -8 \end{bmatrix}$. Now using $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, we get $E_{31}E_{21}A = \begin{bmatrix} 3 & 2 & 6 \\ 0 & 0 & -1 \\ 0 & 1 & -14 \end{bmatrix}$.

   Element $(2, 2)$ cannot be used as a pivot since it is 0, and therefore A has no LU decomposition. To get a $PA = LU$ decomposition, we need to continue and exchange rows 2 and 3. So we get:

$$P_{23}E_{31}E_{21}A = \begin{bmatrix} 3 & 2 & 6 \\ 0 & 1 & -14 \\ 0 & 0 & -1 \end{bmatrix}.$$

U is thus

$$U = \begin{bmatrix} 3 & 2 & 6 \\ 0 & 1 & -14 \\ 0 & 0 & -1 \end{bmatrix},$$

and

$$P = P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

To get L we have to be careful. Right now we have $PCA = U$ where

$$C = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

But $P_{23}C = BP_{23}$ where (B is obtained from C by permuting its rows 2 and 3 and its columns 2 and 3):

$$B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}.$$

The inverse of B is L which is equal to (the inverse of a lower triangular matrix with only one column with nonzero off-diagonal elements is obtained by just switching the sign of the off-diagonal elements; this can be seen by decomposing the matrix into elementary row operations):

$$L = B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

Just to check our calculations, we verify that $PA = LU$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 & 6 \\ 6 & 4 & 11 \\ 3 & 3 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 6 \\ 0 & 1 & -14 \\ 0 & 0 & -1 \end{bmatrix}.$$

4. (6 pts.) If $c = 0$ we have a row of zeroes and then the matrix is certainly singular (impossible to find a matrix B then with $AB = I$). Let's start doing the elimination. After eliminating entries $(2,1)$ and $(3,1)$ we get:

$$\begin{bmatrix} 5 & c & c \\ 0 & c - c^2/5 & c - c^2/5 \\ 0 & 2 - c/5 & 4c/5 \end{bmatrix}.$$

If $c - c^2/5 = 0$, i.e. $c = 0$ or $c = 5$, the second row is zero and A is singular. Assume now that $c \neq 0$ and $c \neq 5$, we can use element $(2,2)$ as a pivot. After elemination, we get:

$$\begin{bmatrix} 5 & c & c \\ 0 & c - c^2/5 & c - c^2/5 \\ 0 & 0 & c - 2 \end{bmatrix}.$$

A is non invertible if and only if some of the diagonal elements of this upper triangular matrix are zero, and this happen precisely if $c = 0$, $c = 5$ or $c = 2$.

5. (4 pts.)

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

6. (7 pts.)

   (a) Yes. One can for example see a lower triangular matrix as a product of elementary row operations matrices and when taking its inverse, we see that it still is the product of lower triangular matrices, which is lower triangular.

   (b) The statement that the LDU factorization is unique if it exists is actually *not true*. It is only true if A is (square and) invertible. Indeed if you take for example the 3×3 matrix equal to 0 then you can take any L and any U provided that you take $D = 0$.

   Assuming that A is non-singular, the proof goes as follows. The matrix $L_1^{-1} L_2 D_2$ is lower triangular (as the product of lower triangular matrices), while $D_1 U_1 U_2^{-1}$ is upper triangular (for the same reason). The only way they can be equal is that they are both diagonal, say equal to D. The fact that $L_1^{-1} L_2 D_2 = D$ can be rewritten as $L_2 = L_1 D D_2^{-1}$; this is the place where we use that A is nonsingular as this implies that D_2 is non-singular. The expression $L_2 = L_1 D D_2^{-1}$ implies that L_2 can be obtained by rescaling the columns of L_1. As both L_1 and L_2 have 1's on the diagonal we have that $L_1 = L_2$. A similar argument shows $U_1 = U_2$. And we have that both D_1 and D_2 are equal to D and thus equal.