18.06 Problem Set 2

SOLUTIONS TO SELECTED PROBLEMS

1. Section 2.5, Problem 25

Answer: \(A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}\); the columns of \(B\) add up to 0, so \(B^{-1}\) does not exist.

2. Section 2.5, Problem 30

Answer: not invertible for \(c = 7\) (equal columns), \(c = 2\) (equal rows), \(c = 0\) (zero column).

3. Section 2.5, Problem 35

Answer:
\[
\begin{bmatrix} I & 0 \\ -C & I \end{bmatrix}; \begin{bmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{bmatrix}; \begin{bmatrix} -D & I \\ I & 0 \end{bmatrix}.
\]

4. Section 2.6, Problem 13

Answer:
\[
\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ b-a & b-a & b-a \\ c-b & c-b \\ d-c \end{bmatrix}.
\]
We need \(a \neq 0, b \neq a, c \neq b, d \neq c\).

5. Section 2.6, Problem 16

Answer:
\[
\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} c = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \rightarrow c = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}; \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \rightarrow x = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.
\]

6. Section 2.6, Problem 19

Answer:
\[
\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = LIU;
\]
\[
\begin{bmatrix}
a & a & 0 \\
a & a+b & b \\
0 & b & b+c
\end{bmatrix}
= \text{(same } L) \begin{bmatrix}
a & b \\
0 & c
\end{bmatrix}\text{ (same } U).}
\]

7. Section 2.6, Problem 28

Answer: \(LU\) will be impossible for \(c = 6\) and \(c = 7\) (a row exchange needed for \(c = 6\)).

8. Section 2.7, Problem 10

Answer: (1,2,3,4), (2,3,1,4), (3,1,2,4), (2,4,3,1), (4,1,3,2), (3,2,4,1),
(4,2,1,3), (1,3,4,2), (1,4,2,3), (2,1,4,3), (3,4,1,2), (4,3,2,1).

9. Section 2.7, Problem 12

Solution: \((Px)^T(Py) = x^T P^T P y = x^T y\) because \(P^T P = I\). In general
\(Px \cdot y = x \cdot P^T y \neq x \cdot Py:\)
\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}
\cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}
\neq \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}
\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.
\]

10. Section 2.7, Problem 16

Answer: \(A^2 - B^2\) and \(ABA\) are symmetric if \(A\) and \(B\) are symmetric.

11. Section 2.7, Problem 19

Answer: (a) \((R^T AR)^T = R^T A^T (R^T)^T = R^T AR\), which is an \(n\) by \(n\) matrix.

(b) The \(j\)-th diagonal entry of \((R^T R)\) is equal to the dot product of the \(j\)-th row of \(R^T\) and the \(j\)-th column of \(R\), i.e. the dot product of the \(j\)-th column of \(R\) with itself, which is positive.

12. Clearly, \(A_\alpha A_\beta = A_{\alpha + \beta}\) since rotating a vector first by angle \(\alpha\) and then by angle \(\beta\) is equivalent to rotating the vector by angle \(\alpha + \beta\). The relation \(A_\alpha A_\beta = A_{\alpha + \beta}\) shows that the set \(G\) of all \(A_\alpha\) for \(0 \leq \alpha < 2\pi\) is closed under multiplication. (Formally, we need to remark that if \(A_\alpha\) and \(A_\beta\) are in \(G\) and \(\alpha + \beta > 2\pi\), then \(0 \leq \alpha + \beta - 2\pi < 2\pi\) and \(A_\alpha A_\beta = A_{\alpha + \beta - 2\pi}\), which is in \(G\).)

The set \(G\) contains the identity matrix \(I = A_0\), and for every \(A_\alpha\) in \(G\), its inverse, namely \(A_{2\pi - \alpha}\), is also in \(G\) (for \(A_\alpha A_{2\pi - \alpha} = A_{2\pi - \alpha} A_\alpha = A_{2\pi} = I\)). Thus \(G\) is a group of matrices.
13. Let \( A = [a_{ij}] \) and \( B = [b_{ij}] \), where \( a_{ij} = a_{ji} \) and \( b_{ij} = b_{ji} \). Set \( AB = [c_{ij}] \) and \( BA = [d_{ij}] \). First, suppose \( AB \) is symmetric. Then

\[
    c_{ij} = c_{ji} = \sum_{k=1}^{n} a_{jk}b_{ki} = \sum_{k=1}^{n} b_{ik}a_{kj} = d_{ij}
\]

for all \( i \) and \( j \), hence \( AB = BA \). On the other hand, if \( AB = BA \), then

\[
    c_{ij} = d_{ij} = \sum_{k=1}^{n} b_{ik}a_{kj} = \sum_{k=1}^{n} a_{jk}b_{ki} = c_{ji}
\]

for all \( i \) and \( j \), hence \( AB \) is symmetric.