Problem 1 Monday 11/6

Do Problem #12 from section 8.3 in your book.

Solution 1

The columns of $A$ must sum to 1, so $A = \begin{bmatrix} .7 & .1 & .2 \\ .1 & .6 & .3 \\ .2 & .3 & .5 \end{bmatrix}$.

Our theory tells us the steady state is the eigenvector with $\lambda = 1$, and sure enough there is one: $x_1 = (1, 1, 1)$ (or any multiple of $x_1$) works.

Why is $x_1 = (1, 1, \ldots, 1)$ a steady state? The entries of $Ax_1$ are the sums of each row. But $A$ is symmetric, so these are the same as the sums of each column, which are 1. So the entries of $Ax_1$ are 1, just like the entries of $x_1$.

Problem 2 Monday 11/6

Of 300 million Americans, 60% own their own home and the other 40% rent. Let’s represent these proportions as a vector: $x = \begin{bmatrix} \text{owners} \\ \text{renters} \end{bmatrix} = \begin{bmatrix} .60 \\ .40 \end{bmatrix}$.

Every year, some proportion of renters will buy a house, and some proportion of homeowners will move to a rental. If these proportions remain constant, we can model this with the “Markov process” $x_{k+1} = Ax_k$ for some 2-by-2 Markov matrix $A$.

Suppose the proportion of homeowners and renters is modeled by this Markov process; it maintains the steady state $x = \begin{bmatrix} .60 \\ .40 \end{bmatrix}$ above; and 90 percent of homeowners in any given year $k$ still own a home in year $k + 1$.

Determine $A$, and estimate how many American renters will buy a home this year.

Solution 2

We know the first column of $A$ (how many homeowners are owners/renters in the following year): $A = \begin{bmatrix} .90 & ? \\ .10 & ? \end{bmatrix}$

We can calculate the second column of $A$ from the steady state $Ax = x$: $\begin{bmatrix} .90 & a \\ .10 & 1 - a \end{bmatrix} \begin{bmatrix} .60 \\ .40 \end{bmatrix} = \begin{bmatrix} .60 \\ .40 \end{bmatrix}$
gives $A = \begin{bmatrix} .90 & .15 \\ .10 & .85 \end{bmatrix}$.

So 15% of American renters, or 6% of Americans, will buy a home, for a total of 18 million new homeowners.

Problem 3 Wednesday 11/8

Find the first three nonzero terms in the Fourier series for the period-2$\pi$ function

$$f(t) = \begin{cases} 1, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$$

Then find the lengths of the original function $\|f(t)\|$ and your three-term approximation $\|g(t)\|$, and the distance $\|f(t) - g(t)\|$ between them.
Solution 3

When we expand \( f(t) \) as a Fourier series, it looks like \( f(t) = a_0 \cdot 1 + a_1 \cos(t) + b_1 \sin(t) + a_2 \cos(2t) + b_2 \sin(2t) + \ldots \). All we have to do is figure out the coefficients \( a_i, b_i \). This is easy, because the basis functions \( 1, \cos(t), \ldots \) are orthogonal — if we take an inner product, all the other terms go away!

So, to find \( a_0 \), we take the inner product with the basis function \( 1 \) —

\[
(f, 1) = \int_0^{2\pi} a_0 \cdot 1 + a_1 \cos(t) + b_1 \sin(t) + \ldots \ dt = \int_0^{2\pi} f(t) \ dt
\]

\[
\int_0^{2\pi} a_0 dt = \int_0^{\pi} dt
\]

\[
2\pi a_0 = \pi
\]

so \( a_0 = 1/2 \).

Similarly for \( a_1 \):

\[
(f, \cos(t)) = \int_0^{2\pi} (a_0 \cdot 1 + a_1 \cos(t) + b_1 \sin(t) + \ldots) \cos(t) \ dt = \int_0^{2\pi} f(t) \cos(t) \ dt
\]

\[
\int_0^{2\pi} a_1 \cos(t) \ dt = \int_0^{\pi} \cos(t) dt
\]

\[
\pi a_1 = 0
\]

so \( a_1 = 0 \); in fact, all the cosine coefficients \( a_k \) are zero.

Similarly for the sine coefficients \( b_k \):

\[
(f, \sin(kt)) = \int_0^{2\pi} (a_0 \cdot 1 + a_1 \cos(t) + b_1 \sin(t) + \ldots) \sin(kt) \ dt = \int_0^{2\pi} f(t) \sin(kt) \ dt
\]

\[
\int_0^{2\pi} a_1 \sin(kt) \ dt = \int_0^{\pi} \sin(kt) dt
\]

\[
\pi b_k = \left[ \frac{-\cos(kt)}{k} \right]_{t=0}^{\pi}
\]

This gives \( b_k = 0 \) if \( k \) is even, and \( b_k = \frac{2}{k\pi} \) if \( k \) is odd.

*(You could also use the book’s formulas to find the coefficients. But this is where they come from.)*

So the first three terms of the Fourier series for \( f(t) \) are

\[
f(t) \approx g(t) = \frac{1}{2} + \frac{2}{\pi} \sin(t) + \frac{2}{3\pi} \sin(3t).
\]

Now we find the lengths.

\[
\|f(t)\|^2 = (f, f) = \int_0^{2\pi} f(t)^2 dt = f_0^\pi dt = \pi, \text{ so } \|f\| = \sqrt{\pi}.
\]

\[
\|g(t)\|^2 = (g, g) = (1/2)^2 + (2/\pi)^2 + (2/3\pi)^2 \text{ (the basis vectors are orthogonal!)}, \text{ so } \|g\| = \sqrt{(1/4) + (40/9\pi^2)} = \sqrt{160 + 9\pi^2}/6\pi.
\]

\[
\|f(t) - g(t)\|^2 = (f - g, f - g) = \int_0^{2\pi} (f(t) - 1/2 - 2/\pi \sin(t) - 2/3\pi \sin(3t))^2 dt = \ldots
\]

You could evaluate that integral, but there’s an easier way: since \( g \) is the orthogonal projection of \( f \) into a subspace, the error \( f - g \) (= \( b_5 \sin(5t) + b_7 \sin(7t) + \ldots \)) is orthogonal to \( g \)!

So \( \|f\|^2 = \|g\|^2 + \|f - g\|^2 \) and \( \|f - g\|^2 = \sqrt{\pi} - \sqrt{160 + 9\pi^2}/6\pi. \)

*(It’s not obvious this is positive (as lengths should be), but it is.)*

Problem 4 Wednesday 11/8

Do Problem #1 from section 10.2 in your book.
Solution 4

You can still find lengths by the Pythagorean theorem (since |a + bi| = √(a² + b²):
\[\|u\| = \sqrt{(1 + 1) + (1 + 1) + (1 + 4)} = \sqrt{9} = 3,\]
and \[\|v\| = \sqrt{(0 + 1) + (0 + 1) + (0 + 1)} = \sqrt{3}.\]
Or take the dot product (don’t forget to conjugate!):
\[\|u\| = \sqrt{u^H u} = \sqrt{(1 - i)(1 = i) + (1 + i)(1 - i) + (1 - 2i)(1 + 2i)} = \sqrt{2 + 2 + 5} = 3,\]
and \[\|v\| = \sqrt{v^H v} = \sqrt{(-i)(+i) + (-i)(+i) + (-i)(+i)} = \sqrt{1 + 1 + 1} = \sqrt{3}.\]
For complex inner products, order matters:
\[u^H v = (1 - i)i + (1 + i)i + (1 - 2i)i = 2 + 3i,\]
but \[v^H u = -i(1 + i) - i(1 - i) - i(1 + 2i) = 2 - 3i!\]
(The difference is that \((u^H v)^H = vu^H\) conjugates \(u\), but \(v^H u\) conjugates \(v\). So the two products are conjugates of each other.)

Problem 5 Wednesday 11/8

Do Problem #2 from section 10.2 in your book.

Solution 5

\[A = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix} \text{ so } A^H = \begin{bmatrix} 1 & -i \\ -i & 1 \\ -i & -i \end{bmatrix}.\]

\[A^H A = \begin{bmatrix} 0 & 2 & i + 1 \\ 2 & 0 & 1 + i \\ 1 - i & -i + 1 & 2 \end{bmatrix}, \]
\[A A^H = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}.\]

Both of these are Hermitian: their conjugate transpose is itself, \(M^H = M^T = M.\)
(There is also a second one that is real (and hence symmetric): this is just a coincidence.)

Problem 6 Wednesday 11/8

Do Problem #17 from section 10.2 in your book.

Solution 6

First find the eigenvalues: \(\lambda^2 - 2(\cos \theta)\lambda + 1 = 0\) has roots \(\lambda = \cos \theta \pm i \sin \theta.\) Notice that both eigenvalues have |\(\lambda| = 1, \) since \(Q\) is orthogonal.

Now find the eigenvectors. For \(\lambda_+ = \cos \theta + i \sin \theta,\) we want a vector \(x\) with \(\begin{bmatrix} -i \sin \theta & -i \sin \theta \\ \sin \theta & -\sin \theta \end{bmatrix} x = 0,\)
such as \(x_+ = \begin{bmatrix} 1 \\ -i \end{bmatrix} .\) Similarly, \(\lambda_- = \cos \theta - i \sin \theta\) has eigenvector \(x_- = \begin{bmatrix} 1 \\ +i \end{bmatrix} .\)
These eigenvectors are automatically orthogonal (that is, \((u_+, u_-) = 1(1) - i(-i) = 1 - 1 = 0),\) but we want the columns of \(U\) to be orthonormal, so we need to divide by the lengths: \(u_+ = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} \)
and \(u_- = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ +i \end{bmatrix}.\)
Then our factorization is
\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta \\
\end{pmatrix}
\begin{pmatrix}
1/\sqrt{2} & 1/\sqrt{2} \\
-1/\sqrt{2} & i/\sqrt{2} \\
\end{pmatrix}
\begin{pmatrix}
\cos \theta + i \sin \theta & 0 \\
0 & \cos \theta - i \sin \theta \\
\end{pmatrix}
\begin{pmatrix}
1/\sqrt{2} & i/\sqrt{2} \\
1/\sqrt{2} & -i/\sqrt{2} \\
\end{pmatrix}
Q \quad U \quad \Lambda \quad U^H
\]

Problem 7 Wednesday 11/8

Do Problem #31 from section 10.2 in your book.

(Hints: $U$ is a matrix, so $U^H U = I$. $\Lambda$ is a matrix, so $\Lambda^H \Lambda$ and $\Lambda \Lambda^H$ are .)

Solution 7

(Answers to hints: $U$ is unitary, so $U^H U = I$. $\Lambda$ is diagonal, so $\Lambda^H \Lambda = \Lambda \Lambda^H$.)

$A^H A = (U \Lambda^H U^H)(U \Lambda U^H) = U \Lambda^H \Lambda U^H$, $AA^H = (U \Lambda U^H)(U \Lambda^H U^H) = U \Lambda \Lambda^H U^H$, and since $\Lambda^H \Lambda = \Lambda \Lambda^H$, these are equal.

Problem 8 Wednesday 11/8

Do Problem #7 from section 10.3 in your book.

Solution 8

Here's one step of the factorization of the Fourier matrix $F_4$:

$$
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & i & i^2 & i^3 \\
1 & i^2 & i^4 & i^5 \\
1 & i^3 & i^6 & i^9
\end{bmatrix}
\begin{bmatrix}
1 & 1 & i & i \\
1 & 1 & i^2 & i^3 \\
1 & 1 & i^4 & i^5 \\
1 & 1 & i^6 & i^7
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & -i \\
1 & -i \\
1 & -i
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
2 & 0 \\
0 & 0
\end{bmatrix}
$$

Now just multiply:

$$
F_4 c =
\begin{bmatrix}
1 & 1 \\
1 & -1 \\
1 & -i \\
1 & -i
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & i^2 \\
1 & 1 \\
1 & i^2
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
2 & 0 \\
0 & 0
\end{bmatrix}
$$

So $c = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ (the frequency-space representation of $f(t) = 1e^{0\pi t} + 0e^{(1/2)\pi t} + e^{i\pi t} + 0e^{(3/2)\pi t}$)

becomes $y = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ (the time-space representation, $\begin{bmatrix} f(0) = 2 \\ f(1) = 0 \\ f(2) = 2 \\ f(3) = 0 \end{bmatrix}$).

Now do the same for $c = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$; we get $c = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} \rightsquigarrow y = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}$.

(In other words, $f(t) = 0e^{0\pi t} + 1e^{(1/2)\pi t} + 0e^{i\pi t} + 1e^{(3/2)\pi t}$ has time-space representation $\begin{bmatrix} f(0) = 2 \\ f(1) = 0 \\ f(2) = -2 \\ f(3) = 0 \end{bmatrix}$.)
Problem 9 Monday 11/13

Do Problem #16 from section 6.3 in your book.

Solution 9

The power series for $e^{kt}$ is $1 + kt + \frac{k^2 t^2}{2} + \frac{k^3 t^3}{6} + \frac{k^4 t^4}{24} + \ldots$

Same thing for $e^{At} = 1 + At + \frac{A^2 t^2}{2} + \frac{A^3 t^3}{6} + \frac{A^4 t^4}{24} + \ldots$

Differentiate: $\frac{d}{dt}e^{At} = A + \frac{A^2 t}{2} + \frac{A^3 t^2}{2} + \frac{A^4 t^3}{6} + \ldots$

which is $A$ times the first four terms above.

(This is almost a proof that $\exp(At)$ is a solution to $u' = Au$ — we should check that all the other terms work, too! Fortunately, it’s easy to see that the pattern holds.)

Problem 10 Monday 11/13

Do Problem #22 from section 6.3 in your book.

Then solve $u' = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} u$ for initial condition $u(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Is the solution stable as $t \to \infty$? Why or why not?

Solution 10

\[
\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}
\]

so

\[
\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -1/2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & -\frac{1}{2} e^t + \frac{1}{2} e^{3t} \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} e^t & e^{3t} \\ -\frac{1}{2} e^t + \frac{1}{2} e^{3t} \end{bmatrix}
\]

At $t = 0$, $\exp(At)$ reduces to $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, as it should.

Solving $u' = Au$:

\[
\begin{bmatrix} e^t & -\frac{1}{2} e^t + \frac{1}{2} e^{3t} \\ 0 & e^{3t} \end{bmatrix} = \begin{bmatrix} e^t & e^{3t} \\ 0 & 2 e^{3t} \end{bmatrix}
\]

As $t \to \infty$ both components go to infinity, so this is not stable.