Problem 1: (15=5+5+5) Take any matrix $A$ of the form $A = B^H C B$, where $B$ has full column rank and $C$ is Hermitian and positive-definite.

(a) Show that $A$ is Hermitian.

Solution For any two matrices $A_1$ and $A_2$, we have $(A_1 A_2)^T = A_2^T A_1^T$ and $A_1 A_2 = \overline{A_2} \overline{A_1}$. Combine them, we have $(A_1 A_2)^H = A_2^H A_1^H$. Similarly, we have for any $A$, $(A^H)^H = A$.

Now back to the problem. Since $C$ is Hermitian, $C^H = C$. So

$$A^H = (B^H C B)^H = B^H C^H (B^H)^H = B^H C B = A,$$

that is, $A$ is Hermitian.

(b) Show that $A$ is positive-definite by showing that $x \cdot (Ax) > 0$ for $x \neq 0$ (hint: very similar to how we showed that $B^H B$ is positive-definite, in class).

Solution Suppose $x \neq 0$. Since $B$ has full column rank, its nullspace only contains $0$. So $Bx \neq 0$. So we have

$$x \cdot (Ax) = x \cdot (B^H C B x) = (B x) \cdot C (B x) > 0,$$

the last inequality comes from the fact that $C$ is a positive definite matrix.

(c) Show that $A = D^H D$ for some $D$ with full column rank. (Hint: use $\sqrt{C}$ as defined in an earlier problem set.)

Solution We first recall the definition of $\sqrt{C}$. Since $C$ is Hermitian, it can be decomposed to $C = Q^H \Lambda Q$, where $Q$ is unitary and $\Lambda$ is diagonal whose diagonal entries are eigenvalues of $C$. Since $C$ is positive definite, the diagonal entries of $\Lambda$ are all positive, thus the square root matrix $\sqrt{C} = Q^H \sqrt{\Lambda} Q$ is well-defined, as we have seen in the earlier problem set. Notice that in particular we have $\sqrt{\Lambda}^H = \sqrt{\Lambda}$, thus $\Lambda = (\sqrt{\Lambda})^2 = \sqrt{\Lambda^H} \sqrt{\Lambda}$.

Now we have

$$A = B^H C B = B^H Q^H \Lambda Q B = B^H Q^H \sqrt{\Lambda}^H \sqrt{\Lambda} Q B.$$

If we denote $D = \sqrt{\Lambda} Q B$, we get immediately $A = D^H D$. Finally since both $\sqrt{\Lambda}$ and $Q$ are nonsingular, and $B$ is of full column rank, we see that $D$ is of full column rank.
Problem 2: Consider Poisson’s equation \( \frac{d^2 f}{dx^2} = g(x) \) given and you want to find \( f(x) \). In lecture, we studied this for the case where \( f \) (and \( g \)) belongs to the space of real functions on \( x \in [0, 1] \) with \( f(0) = f(1) = 0 \): we solved it by expanding \( f \) and \( g \) in Fourier sine series and then inverting each eigenvalue. Now, you should see what happens in the space of functions with zero slope at the boundaries \( [f'(0) = f'(1) = 0] \), where the eigenfunctions of \( \frac{d^2}{dx^2} \) gave the Fourier cosine series.

(a) What is the null space of \( \frac{d^2}{dx^2} \)? (Note that you should only consider functions in the vector space, i.e. with zero slope at \( x = 0 \) and \( x = 1 \).)

**Solution** Suppose

\[
\frac{d^2 f}{dx^2} = \frac{d}{dx} \left( \frac{df}{dx} \right) = 0,
\]

then \( \frac{df}{dx} = a \) is constant. It follows that

\[
f(x) = ax + b
\]

is some linear function. Now since \( f'(0) = f'(1) = 0 \) and \( f'(x) = a \), we get \( a = 0 \). So \( f(x) = b \) is constant function. So the nullspace of \( \frac{d^2}{dx^2} \) consists all the constant functions. (This is a one dimensional subspace.)

(b) What is the column space of \( \frac{d^2}{dx^2} \), in terms of the Fourier cosine series? That is, if \( \frac{d^2 f}{dx^2} = g(x) \), and you write out the cosine series of \( g(x) \), what are the possible coefficients? (Hint: start with the cosine series of \( f(x) \), and see what happens to it when you take the second derivative—what possible right-hand-sides can you get?)

**Solution** Suppose

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n \pi x).
\]

Then

\[
\frac{df}{dx} = -\sum_{n=1}^{\infty} n \pi a_n \sin(n \pi x)
\]

and thus

\[
\frac{d^2 f}{dx^2} = -\sum_{n=1}^{\infty} n^2 \pi^2 a_n \cos(n \pi x).
\]

So the column space of \( \frac{d^2}{dx^2} \) consists all functions whose Fourier cosine series has vanishing first \( (0^{th}) \) order coefficient \( a_0 \), which by definition is \( 2\int_{0}^{1} f(t) dt \). We conclude that the column space consists of functions \( f(x) \) with zero integral, i.e. functions whose average value is 0.
(c) Suppose that \( g(x) \) is the function \( g(x) = 1 \) for \( x < 1/2 \) and \( g(x) = -1 \) for \( x \geq 1/2 \). Find the cosine series of \( g(x) \), using the cosine-series formulas:

\[
g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x)
\]

\[
a_n = 2 \int_{0}^{1} g(x) \cos(n\pi x) \, dx.
\]

[The \( n = 0 \) term has a 1/2 factor in the first formula so that the second formula works for all \( n \). The reason for the difference is just a matter of normalization: \( \| \cos(0\pi x) \|^2 = 1 \), but \( \| \cos(n\pi x) \|^2 = 1/2 \) for \( n > 0 \).] Hint: you should find that \( a_n = 0 \) for even \( n \).

[Solution] We have

\[
a_0 = 2 \int_{0}^{1} g(x) \, dx = 2 \left( \int_{0}^{1/2} dx + \int_{1/2}^{1} (-1) \, dx \right) = 0,
\]

and for \( n \geq 1 \),

\[
a_n = 2 \int_{0}^{1} g(x) \cos(n\pi x) \, dx
\]

\[
= 2 \int_{0}^{1/2} \cos(n\pi x) \, dx - 2 \int_{1/2}^{1} \cos(n\pi x) \, dx
\]

\[
= \frac{2}{n\pi} \sin(n\pi x) \bigg|_{0}^{1/2} - \frac{2}{n\pi} \sin(n\pi x) \bigg|_{1/2}^{1}
\]

\[
= \frac{4}{n\pi} \sin(n\pi/2).
\]

Notice that \( \sin(n\pi/2) \) equals 0 when \( n \) is even, equals 1 when \( n \) is of the form \( 4k + 1 \), and equals \(-1\) when \( n \) is of the form \( 4k + 3 \). We finally find the cosine series of \( g \),

\[
g(x) = \sum_{k=1}^{\infty} \left( \frac{4}{(4k + 1)\pi} \cos((4k + 1)\pi x) - \frac{4}{(4k + 3)\pi} \cos((4k + 3)\pi x) \right)
\]

\[
= \frac{4}{\pi} \cos(\pi x) - \frac{4}{3\pi} \cos(3\pi x) + \frac{4}{5\pi} \cos(5\pi x) - \frac{4}{7\pi} \cos(7\pi x) + \cdots
\]
(d) Verify that $g(x)$ from (c) is in the column space from (b). Using your answer from (c), find the cosine series for $f(x)$ to satisfy Poisson’s equation. $f(x)$ should be the sum of a particular solution plus an arbitrary nullspace solution, using your answer to (a).

**Solution** Since $a_0 = 0$, $g(x)$ lies in the column space of the operator $d^2/dx^2$.

To find a particular solution $f(x)$, we suppose $f(x)$ has the cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x).$$

We have seen above that

$$\frac{d^2 f}{dx^2} = -\sum_{n=1}^{\infty} n^2 \pi^2 a_n \cos n\pi x.$$

Compare its coefficients with the coefficients of the cosine series of $g$ above, we conclude that $a_n = 0$ for even $n \geq 2$, $a_n = -\frac{4}{n^2 \pi^2}$ for odd $n$ of the form $4k + 1$, and $a_n = \frac{4}{n^2 \pi^2}$ for odd $n$ of the form $4k + 3$. There is no restriction on $a_0$, so it can be arbitrary constant, which corresponds to arbitrary nullspace solution. (The Fourier series method is easy to solve ODE, since inverting $d^2/dx^2$ corresponds to simply multiplying each eigenfunction by the inverse of the eigenvalues.)

(e) In Matlab, plot the first four nonzero terms of your $g(x)$ cosine series, and then plot the first 8 nonzero terms—verify that the cosine series is converging to $g(x)$ (except right at the point of the discontinuity). For example, if you put the coefficients in the variables $a0$, $a1$, and so on (e.g. $a0 = 1.234/\pi$), then you can plot the first four terms of the Fourier cosine series with the command:

```matlab
fplot(@(x) a0/2 + a1*cos(pi*x) + a2*cos(2*pi*x) + a3*cos(3*pi*x), [0,1])
```

**Solution** The Inputs are

```matlab
>> fplot(@(x) 4/pi*cos(pi*x) -4/(3*pi)*cos(3*pi*x) + 4/(5*pi)*cos(5*pi*x) - 4/(7*pi)*cos(7*pi*x), [0,1])
>> fplot(@(x) 4/pi*cos(pi*x) -4/(3*pi)*cos(3*pi*x) + 4/(5*pi)*cos(5*pi*x) - 4/(7*pi)*cos(7*pi*x) + 4/(9*pi)*cos(9*pi*x) - 4/(11*pi)*cos(11*pi*x) + 4/(13*pi)*cos(13*pi*x) - 4/(15*pi)*cos(15*pi*x), [0,1])
```

**Outputs**
Figure 1: 4 nonzero terms of $g$

Figure 2: 8 nonzero terms of $g$
(f) As in (e), but plot the first 4 and 8 non-zero terms of your solution $f(x)$ from (d) [just pick some value for the nullspace part of the solution]. Which series converges faster, the one for $f$ or the one for $g$?

Solution We pick the arbitrary constant $a_0$ to be zero. The codes:

```matlab
>> fplot(@(x) -4/(pi^3)*cos(pi*x) +4/(3^3*pi^3)*cos(3*pi*x) - 4/(5^3*pi^3)*cos(5*pi*x) + 4/(7^3*pi^3)*cos(7*pi*x), [0,1])
```  

```matlab
>> fplot(@(x) -4/(pi^3)*cos(pi*x) +4/(3^3*pi^3)*cos(3*pi*x) - 4/(5^3*pi^3)*cos(5*pi*x) + 4/(7^3*pi^3)*cos(7*pi*x) - 4/(9^3*pi^3)*cos(9*pi*x) + 4/(11^3*pi^3)*cos(11*pi*x) - 4/(13^3*pi^3)*cos(13*pi*x) + 4/(15^3*pi^3)*cos(15*pi*x), [0,1])
```

Outputs

Figure 3: 4 nonzero terms of $f$

It turns out that the cosine series for $f$ converges much faster. This is true in general, since the denominator of each term in the cosine series of $f$ is much bigger than the corresponding term of $g$. (There is an extra $n^2$ term)
Problem 3: (18=13+5) (1) Follow the steps in problem 13 on page 350 to show that $A^T$ is always similar to $A$.

**Solution**

- **Step 1:** $J_i$ is similar to $J_i^T$.

  Recall that any Jordan block is of the form
  \[
  J_i = \begin{pmatrix}
  \lambda_i & 1 & 0 & \cdots & 0 \\
  0 & \lambda_i & 1 & \cdots & 0 \\
  0 & 0 & \lambda_i & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & \lambda_i
  \end{pmatrix}.
  \]

  Take $M_i$ to be the anti-diagonal matrix whose anti-diagonal elements are 1, and whose size is the same as $J_i$, i.e.
  \[
  M_i = \begin{pmatrix}
  0 & \cdots & 0 & 0 & 1 \\
  0 & \cdots & 0 & 1 & 0 \\
  0 & \cdots & 1 & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots \\
  1 & \cdots & 0 & 0 & 0
  \end{pmatrix}.
  \]

  Then one can easily check that $M_i^{-1} = M_i$. Notice that multiply to the left by $M_i$ just changes the rows from 1, 2, \cdots, $n$ to $n$, $n-1$, \cdots, 1, and multiply to the right by $M_i$ just
changes the columns from $1, 2, \ldots, n$ to $n, n-1, \ldots, 1$. So it follows

$$M_i^{-1}J_iM_i = M_iJ_iM_i = J_i^T.$$ 

\[ \begin{align*}
\text{Step 2: Now given the Jordan form} \\
J &= \begin{pmatrix}
J_1 & 0 & \cdots & 0 \\
0 & J_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & J_r
\end{pmatrix}, \\
\text{we take} \\
M &= \begin{pmatrix}
M_1 & 0 & \cdots & 0 \\
0 & M_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & M_r
\end{pmatrix},
\end{align*} \]

it follows from part one that $M_i^{-1}JM_i = J_i^T$.

\[ \begin{align*}
\text{Step 3: Suppose} \ A = PJP^{-1}, \text{where} \ J \text{is Jordan form above. Take} \ Q = MP^T, \text{then} \\
A^T &= (P^{-1})^TJ^TP^T = (P^T)^{-1}M^{-1}JMP^T = Q^{-1}JQ,
\end{align*} \]

so $A^T$ is similar to $A$.

(2) Is $A^H$ always similar to $A$? Justify your conclusion.

\[ \text{[Solution] No.} \]

Since the eigenvalues of $A^T$ are the same eigenvalues of $A$, we see that the eigenvalues of $A^H$ are the conjugate of the eigenvalues of $A$. So in general $A^H$ has different eigenvalues of $A$, and thus they are not similar.
Problem 4: \((16=4+4+4+4)\) Let \(A = uv^T\) be any rank-1 matrix.

1) What is the dimension of \(N(A)\)? What is \(C(A)\)?

**Solution** Since \(A\) is rank-1 matrix, the dimension of \(N(A)\) is \(n-1\).

(For any two matrices \(A\) and \(B\), the column space of \(AB\) always lies in the column space of \(A\).) Since the column space of \(A\) is one dimensional, and is contained in the line spanned by \(u\), we see that the column space of \(A\) is exactly the line spanned by \(u\).

2) Find all eigenvalues of \(A\), assuming that \(u\) and \(v\) both have \(n\) components so that \(A\) is square.

**Solution** Since \(N(A)\) is \(n-1\) dimensional, \(0\) is an eigenvalue of \(A\) with multiplicity \(n-1\).

Since \(C(A)\) is one dimensional and contains \(u\), the vector \(u\) must be an eigenvector. Now

\[ Au = (uv^T)u = u(v^Tu) = (v^Tu)u, \]

so the corresponding eigenvalue (the last one) is \(v^Tu\). (However, if \(v^Tu = 0\), then all eigenvalues are 0, AND there are only \(n-1\) eigenvectors, in which case the matrix is not diagonalizable.)

3) What are the singular values of \(A\)? What is an SVD for \(A\)?

**Solution** Since \(AA^T = uv^Tu u^T = (v^Tv)uu^T\), so the eigenvalues of \(AA^T\) are \(0\) (of multiplicity \(n-1\)) and \((v^Tv)(u^Tu)\) (which is nonzero). Thus the singular values of \(A\) is \(0, \ldots, 0, \sqrt{(v^Tv)}(u^Tu)\).

Now SVD for \(A\) is \(U\Sigma V\), where \(U\) is unitary whose first \(n-1\) columns are an orthonormal basis of \(N(A)\) (= the hyperplane whose normal vector is \(u\)), and the last column is \(\hat{u}\), the normalization of \(u\), and \(V\) has the same description, replacing \(u\) by \(v\).

4) Construct a rank-1 matrix \(A\) so that \(A(1,0,1)^T = (2,1)^T\).

**Solution** \(A\) must be a \(2 \times 3\) matrix. Thus \(A = uv^T\), with \(u\) a 2-vector, and \(v\) a 3-vector. Since \((2,1)^T\) lies in the column space which is one dimensional, we see from above that it is eigenvector which is a multiple of \(u\). We can take \(u = (2,1)^T\). Now from \(A(1,0,1)^T = (2,1)^T\) we get \(v^T(1,0,1)^T = 1\). We can take, for example, \(v = (1,0,0)^T\). Thus we may take \(A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \).

(The general form of \(v\) is \((a,b,1-a)^T\), and the general \(A\) is \(A = \begin{pmatrix} 2a & 2b & 2-2a \\ a & b & 1-a \end{pmatrix} \).)
Problem 5: (16=4+4+4+4) (1) Find the SVD for \( A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \).

Solution We have \( A^T A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \), whose eigenvalues are \( \lambda_1 = 3 \) and \( \lambda_2 = 1 \), and corresponding eigenvectors, after normalization, are \( v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \) and \( v_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \).

It follows that \( u_1 = \frac{1}{\sqrt{3}} A v_1 = \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix} \), \( u_2 = A v_2 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} \).

The vector \( u_3 \) is the unit vector in the nullspace of \( A^T \), which is \( u_3 = \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \). So the SVD for \( A \) is

\[
A = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}.
\]

(2) Find the pseudoinverse \( B \) of \( A \).

Solution The pseudoinverse \( B \) of \( A \) is

\[
B = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}
\]

\[
= \begin{pmatrix} 2/3 & 1/3 & -1/3 \\ -1/3 & 1/3 & 2/3 \end{pmatrix}.
\]

(3) Compute \( AB, BA, ABA, \) and \( BAB \).

Solution Just straightforward computation:

\[
AB = \begin{pmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{pmatrix}, \quad BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

and thus \( ABA = A(BA) = AI = A, BAB = (BA)B = IB = B \).
(4) What are $ABA$ and $BAB$ for a general matrix $A$ and its pseudoinverse $B$, given the definitions of the SVD and pseudoinverse?

**Solution** We will always have $ABA = A$ and $BAB = B$. In fact, we have

$$
\Sigma^+ \Sigma = \begin{pmatrix}
\frac{1}{\sigma_1} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \frac{1}{\sigma_r}
\end{pmatrix}
\begin{pmatrix}
\sigma_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \sigma_r
\end{pmatrix}
= \begin{pmatrix}
1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1
\end{pmatrix},
$$

it follows $\Sigma \Sigma^+ = \Sigma$ and $\Sigma^+ \Sigma \Sigma^+ = \Sigma^+$. Thus

$$ABA = U \Sigma V^H \Sigma^+ U^H U \Sigma V^H = U \Sigma \Sigma^+ \Sigma V^H = U \Sigma V^H = A$$

and

$$ABA = V \Sigma^+ U^H U \Sigma V^H V \Sigma^+ U^H = V \Sigma^+ \Sigma \Sigma^+ U^H = V \Sigma^+ U^H = B.$$

**Problem 6:** (10=5+5) Given the SVD $A = U \Sigma V^H$.

(1) What is the SVD of $A^H$ and $A^{-1}$ (assuming $A$ is invertible)?

**Solution** For $A^H$ we have $A^H = V \Sigma^H U^H = V \Sigma U^H$.

Suppose $A$ is invertible, then it is square matrix, and $\Sigma$ is invertible diagonal matrix. So the SVD of $A^{-1}$ is $A^{-1} = V \Sigma^{-1} U^H$.

(2) If the QR decomposition of $A$ is $A = QR$, what is the SVD for $R$?

**Solution** If $A$ is a square matrix, then $A = QR = U \Sigma V^H$, so it follows $R = (Q^H U) \Sigma V^H$, which is the SVD for $R$.

If $A$ is an $m \times n$ non-square with full column rank (thus $m \geq n$), then $A = QR$ is also defined to be the result of Gram-Schmidt, in which case $Q$ is $m \times n$ matrix whose columns are orthonormal vectors, and $R$ is $n \times n$ and invertible. In this case, given the SVD $A = U \Sigma V$, we may take only the first $n$ columns of $U$, and the upper-left $n \times n$ corner of $\Sigma$, corresponding to a basis for $C(A)$ only, then apply above arguments.