SOLUTIONS TO PSET 10

Problem 1. The wavelet basis is $w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $w_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $w_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $w_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. If $e = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, then $e = (1/4)(w_1 + w_2 + 2w_3)$, and $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = w_3 + w_4$.

Problem 2. We have that $b_1v_1 + ... + b_nv_n = Vb = c_1w_1 + ... + c_nw_n = We$. Therefore $b = V^{-1}We$, and so $V^{-1}W$ is $M$, the change of basis matrix.

Problem 3. Well, $W^* = (W^{-1})^*$, and this implies that $(W^*)^* = ((W^{-1})^*)^* = (W^{-1})^*$ = $W$ (use that inverse and transpose commute).

Problem 4. (5 points each)

1. $A = \begin{pmatrix} 3 & 0 \\ 1 & 3 \\ -1 & 1 \end{pmatrix}$. Therefore, $AA' = \begin{pmatrix} 3 & 0 \\ 1 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 3 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 8 \\ 8 & 10 \end{pmatrix}$. We find the eigenvalues by computing $\det \left( \begin{array}{cc} 10 - \lambda & 8 \\ 8 & 10 - \lambda \end{array} \right) = (10 - \lambda)^2 - 64 = 100 + \lambda^2 - 20\lambda - 64 = \lambda^2 - 20\lambda + 36 = (\lambda - 18)(\lambda - 2)$. So the singular values are $\sqrt{18}$ and $\sqrt{2}$. For the eigenvalue $\lambda = 18$, we find the unit eigenvector by solving $\begin{pmatrix} -8 & 8 \\ 8 & -8 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$, and so we get $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$.

2. With the same $A$, $AA' = \begin{pmatrix} 3 & 0 \\ 1 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 3 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 18 & 0 \\ 0 & 2 \end{pmatrix}$, so we see the same eigenvalues, and arrive at eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. From this we obtain the SVD for $A$ which reads $A = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \sqrt{18} \\ 0 \\ \sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$.

Problem 5. Suppose $A$ is $n \times n$. If $\det(A) = 0$, then it must be that $rk(A) < n$. But we have that $rk(A^+) = rk(A)$ (this follows from the definition of $A^+ = V\Sigma^+U^T$; the rank of $A$ is the number of nonzero singular values, which is also the number of nonzero entries of $\Sigma^+$). Thus $A^+$ doesn’t have full rank, and so it isn’t invertible, and so $\det(A^+) = 0$.

Problem 6. Let $x$ be any solution to $A^TAx = A^Tb$. Then as $A^TAx^+ = A^Tb$, we see that $A^TA(x - x^+) = A^Tb - A^Tb = 0$. Thus $x - x^+$ is in $N(A^TA) = N(A)$. Now, as $N(A)^\perp = \text{Row}(A)$, the equality $||x||^2 = ||x^+||^2 + ||x - x^+||^2$ will follow if we can show that $x^+ = A^+b$ is in $\text{Row}(A)$. But recall the definition of $A^+$: we had that $A^+u_i = \sigma_i^{-1}v_i$ (for $1 \leq i \leq r$), and $A^+u_i = 0$ for $i > r$. Thus the image $A^+$ is exactly $\text{span}\{v_1, ..., v_r\} = \text{Row}(A)$.

Problem 7. Well, $AA^+$ is the projection onto the column space of $A$, while $A^+A$ is the projection onto the rowspace. Therefore, if $b = p + e$ is the decomposition of $b$ into its
column space and left nullspace part, then $A A^+ p = p$ while $A A^+ e = 0$. Similarly, $A^+ A x_r = x_r$, while $A^+ A x_n = 0$. 