SOLUTIONS TO PSET 9

Problem 1. (5 points each)
1. B = M⁻¹AM and C = N⁻¹BN imply C = N⁻¹M⁻¹AMN = (MN)⁻¹A(MN). If B is similar to A and C is similar to B, then A is similar to C.
2. F⁻¹AF = C = G⁻¹BG, so B = GF⁻¹AFG⁻¹ = (FG⁻¹)⁻¹A(FG⁻¹). If C is similar to A and also to B, then A and B are similar.

Problem 2. Let M be a 4 × 4 matrix. Then JM = \[
\begin{pmatrix}
m_{21} & m_{22} & m_{32} & m_{42} \\
0 & 0 & 0 & 0 \\
m_{41} & m_{42} & m_{43} & m_{44} \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
while
MK = \[
\begin{pmatrix}
0 & m_{11} & m_{12} & 0 \\
0 & m_{21} & m_{22} & 0 \\
0 & m_{31} & m_{32} & 0 \\
0 & m_{41} & m_{42} & 0
\end{pmatrix}
\]. If JM = MK, then we conclude from comparing these two matrices that m_{11} = m_{22} = 0, and m_{21} = 0, and m_{31} = m_{42} = 0, and m_{41} = 0. Thus det(M) = 0 and M is not invertible as required.

Problem 3. (2.5 points each)
a) True. If A = MBM⁻¹ with B invertible, then det(A) = det(M)det(B)det(M⁻¹) = det(B) ≠ 0.
b) False. \[
\begin{pmatrix}
1 & 1 \\
2 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 3 \\
2 & -1
\end{pmatrix}
= \begin{pmatrix}
10 & -3 \\
15 & -5
\end{pmatrix}.
\]
c) False. We know (problem 13) that A and A’ are similar. So just choose a nonzero skew-symmetric matrix.
d) True. If A is similar to A + I, then tr(A) = tr(A + I) = tr(a) + n, which is impossible.

Problem 4. We have that \{v_i\} and \{u_i\} are orthonormal bases in \(R^m\). We want A such that Av_i = u_i. If we let V be the matrix whose columns are the v_i, and U the matrix whose columns are the u_i, then what we are asking for is AV = U, or, equivalently, A = UV^T.

Problem 5. Here, we suppose that the \(m \times n\) matrix A has orthogonal columns, labelled \(\{w_i\}\), with lengths \(\{\sigma_i\}\). This tells us that \(A' A = \Lambda\), where \(\Lambda\) is the diagonal matrix with eigenvalues \(\sigma_i^2\). Thus \(V = I\). So the SVD reads \(A = U\Sigma\). So we can let \(\Sigma\) be the \(m \times n\) matrix whose first \(n\) diagonal elements are the \(\sigma_i\) and all of whose other elements are 0 (note that \(m \geq n\) because the columns of A are orthogonal, hence independent), and we can let U be the orthogonal \(m \times m\) matrix whose first \(n\) columns are \((1/\sqrt{\sigma_i})w_i\), and the rest of whose columns form an orthonormal basis for the left nullspace of A.

Problem 6. (5 points each)
1. We have \(T: M_2(\mathbb{R}) \rightarrow M_2(\mathbb{R})\) given by \(T(M) = AM\). Then \(T(M_1 + M_2) = A(M_1 + M_2) = AM_1 + AM_2 = T(M_1) + T(M_2)\). \(T(\lambda M) = A(\lambda M) = \lambda (AM) = \lambda T(M)\); so \(T\) is linear.
2. Suppose \(A = \begin{pmatrix}1 & 2 \\ 3 & 5\end{pmatrix}\). Then \(det(A) = -1\), so \(A\) is invertible. Now, \(T(M) = AM = 0\) implies \(0 = A^{-1}(AM) = M\). Further, given \(B\), \(T(A^{-1}B) = A(A^{-1}B) = B\).

Problem 7. (2 points for 15, 2 each for 17)
1. Now put \( A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \). Then \( \det(A) = 0 \), so \( A \) is not invertible. Thus \( T(M) = AM = I \)
is impossible. Further, \( \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix} = 0 \).

2. a) True. \( T^2(A) = TT(A) = T(A') = (A')' = A \).
b) True. \( T \) is invertible (it is its own inverse, by part a), so \( \ker(T) = 0 \).
c) True. For any \( B, B = T(B') \).
d) False. This is just the skew-symmetry condition.

Problem 8. Clearly we have 
\[ B = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \] as \( S(1) = 0, S(x) = 0, S(x^2) = 2, S(x^3) = 6x \).

Problem 9. (5 points each)
1. The matrix for \( T \) is 
\( \begin{pmatrix} T(v_1) & T(v_2) & T(v_3) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \). Further, \( T(v_1 - v_2) = w_1 + w_2 + w_3 - (w_2 + w_3) = w_1 \).
2. Well, \( T^{-1}(w_3) = v_3 \) clearly. Next, \( T^{-1}(w_2 + w_3) = v_2 \), so \( T^{-1}(w_2) = v_2 - v_3 \). Finally, \( v_1 = T^{-1}(w_1 + w_2 + w_3) = T^{-1}(w_1) + v_2 - v_3 + v_3 = T^{-1}(w_1) + v_2 \), so \( T^{-1}(w_1) = v_1 - v_2 \). Thus \( A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \). \( Tv = 0 \) only happens when \( v = 0 \), because \( T \) is invertible.

Problem 10. We have \( A = QR \). Now, any invertible matrix \( B \) can be interpreted as the c.o.b. matrix from the basis which consists of columns of \( B \) to the standard basis (this is his definition of c.o.b. matrix in the text). Thus, \( A \) is the c.o.b. matrix from the basis \( \{a_1, a_2, a_3\} \) to the standard basis and \( Q \) is the c.o.b. from the basis \( \{q_1, q_2, q_3\} \) to the standard basis. So \( Q^{-1} \) is the c.o.b. matrix from the standard basis to the basis \( \{q_1, q_2, q_3\} \). So \( R = Q^{-1}a \) is the c.o.b. matrix from the basis \( \{a_1, a_2, a_3\} \) to the basis \( \{q_1, q_2, q_3\} \).