QUIZ 2 SOLUTIONS

1. (10 points). \( \det(-A') = (-1)^{1000} \det(A') = \det(A) \).

2. a) (10 points). The projection matrix of a matrix \( A \) is \( P = A(A'A)^{-1}A' \). So the projection matrix of \( QA \) is \( (QA)(A'QA)^{-1}A'Q = QA(A'A)^{-1}A'Q = PQQ' \) where we have used that \( QQ' = I \).

b) (10 points). By definition \( c - Pc \) is orthogonal to the space \( \text{span} \{ a, b \} = \text{span} \{ q_1, q_2 \} \). So we can choose \( q_3 = (c - Pc)/||c - Pc|| \).

c) (10 points). Let \( s_1, s_2, s_3 \) denote the rows of \( Q \), and \( r_1, r_2, r_3 \) denote the columns of \( R \). Then \( c = \begin{bmatrix} s_1 \cdot r_3 \\ s_2 \cdot r_3 \\ s_3 \cdot r_3 \end{bmatrix} = Qr_3 \). Since orthogonal matrices preserve the lengths of vectors, this implies \( ||c|| = ||r_3|| \).

3. (15 points). Well, the matrix \( uu^t = (u_iu_j)_{1 \leq i, j \leq n} \); so \( I + t uu^t = (\delta_{ij} + tu_iu_j) = Q \). In particular, this matrix is symmetric, so the orthogonality condition reduces to \( Q^2 = I \). Writing this condition out gives \( I = (I + tu_iu_j^t)(I + tu_iu_j) = I + 2tu_iu_j^t + t^2(u_iu_j^t)^2 \), or equivalently \( t(2uu^t + t(uu^t)^2) = 0 \). But now we have that \( (uu^t)^2 = uu^t uu^t = uu^t \) because \( uu^t \) is a column of \( R \) with length 1. So our equation becomes \( t(2 + t)(uu^t)^2 = 0 \). Clearly \( t = 0 \) and \( t = -2 \) are the solutions.

4. (15 points). We have that \( C + Dt + (1 - E)t = (C + E) + (D - E)t \). Thus we see that \( E \) is a free variable: it is not uniquely determined, and in fact can take any value. Given this, just write down the usual least squares equations but treat \( C + E \) and \( D - E \) as your variables: the matrix \( A \) has two columns: the first consists of \( n \) 1’s, the second is the vector \( (t_i) \). Then solve \( A'A \begin{bmatrix} C + E \\ D - E \end{bmatrix} = A'b \).

5. a) (15 points). Yes. As \( A \) is invertible, it’s column space is the full space \( \mathbb{R}^n \). The same is true of \( A^{-1} \).

b) (15 points). No. consider \( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \). It has nonzero column space, but its square is 0.