18.06 Professor Postnikov Quiz 3 November 23, 2009

Your PRINTED name is: SOLUTIONS

Please circle your recitation:

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Grading

1

2

3

Total:

Problem 1. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$.

(A) Find the eigenvalues and the eigenvectors of A.

(B) Solve the differential equation $\frac{d\,\mathbf{u}(t)}{dt} = A\,\mathbf{u}(t)$ with the initial condition $\mathbf{u}(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

(C) Find a symmetric matrix B which is similar to A.

(D) Find the singular values σ_1 and σ_2 of A.

(A): $\det(A - \lambda I) = (1-\lambda)(-1-\lambda)$. Eigenvalues are $1, -1$

$\begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} \rightsquigarrow$ eigenvector for 1 is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow$ eigenvector for -1 is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

(B) $\frac{d\,u(t)}{dt} = A u(t) \implies u(t) = C e^{t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + D e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$u(0) = C \begin{pmatrix} 1 \\ 0 \end{pmatrix} + D \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \implies \begin{matrix} C = 1 \\ D = -1 \end{matrix}$

$\boxed{\therefore\ u(t) = e^{t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$

(C) A is diagonalizable (2 distinct eigenvals / vectors)

So A is similar to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(D) $A^T A = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $\det(A^T A - \lambda I) = (1-\lambda)(2-\lambda) - 1 = 0$

$= \lambda^2 - 3\lambda + 1$

$\lambda = \frac{3 \pm \sqrt{9-4}}{2}$

$\sigma_1, \sigma_2 = \sqrt{\frac{3 \pm \sqrt{5}}{2}}$

Problem 2. Consider the matrix

$$A = \begin{pmatrix} 1 & t & 0 \\ t & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix},$$

which depends on a parameter t.

(A) Find all values of the parameter t when the matrix A is positive definite.

(B) Suppose that $t = 0$. Find a 3×3 matrix R such that $A = R^T R$.

(C) Suppose that $t = 0$. Verify directly that A satisfies the energy-based definition of a positive definite matrix, as follows. For a vector $\mathbf{x} = (x, y, z)^T$, write out $\mathbf{x}^T A \mathbf{x}$; show that this can be written as a sum of squares; and deduce that $\mathbf{x}^T A \mathbf{x} > 0$ for any non-zero $\mathbf{x}$.

(A) $1 > 0$, $1 - t^2 > 0$, $1 \cdot (2-1) - t \cdot (2t) = 1 - 2t^2 > 0$

$\Rightarrow t^2 < \frac{1}{2}$ or $-\sqrt{\frac{1}{2}} < t < \sqrt{\frac{1}{2}}$.

(B) From problem 1, we saw $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

So take

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

(C) $x^T A x = x^T R^T R x = (Rx)^T Rx = \|Rx\| > 0 \quad \forall x$

because R is nonsingular (has 3 pivots)

Problem 3. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$.

(A) Indicate which of the following statements are true and which are false:

(1) A is symmetric; (2) A is orthogonal;

(3) A is invertible; (4) $\frac{1}{3}A$ is a Markov matrix

(B) Find the eigenvalues and the eigenvectors of A. (Hint: Part (A) might help you.)

(C) Find an orthogonal matrix Q and a diagonal matrix Λ such that $A = Q\Lambda Q^T$.

(D) Calculate the limit $\mathbf{u}_\infty$ of $\mathbf{u}_k = (\frac{1}{3}A)^k \mathbf{u}_0$ as $k \to \infty$, for $\mathbf{u}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

(A) (1) Yes!

(2) No: $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \neq 0$, and vectors aren't unit length.

(3) No: $2 \cdot \text{row } 1 = \text{row } 2 + \text{row } 3$

(4) Yes! Colums all add to 3.

(B) A is singular, so 0 is an eigenvalue.

$\frac{1}{3}A$ is Markov $\Rightarrow$ 1 is eigenval of $\frac{1}{3}A \Longrightarrow 3$ is eigenval of A.

$tr(A) = 1+2+2 = 3+0+\lambda \Longrightarrow \lambda = 2$ is other eigenvalue.

$0 \rightsquigarrow \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ $2: \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ $3: \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ (row sums are = 3)

(C) Since A is symmetric, eigenvects are orthogonal: must normalize!

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(c) continued:

$$Q = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} \qquad \Lambda = \begin{pmatrix} 0 & & \\ & 2 & \\ & & 3 \end{pmatrix}$$

D. Calculate lim u_∞ : $u_k = \left(\frac{1}{3}A\right)^k u_0 \qquad u_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

$$\tfrac{1}{3}A = Q\,\tfrac{1}{3}\Lambda Q^T$$

$$\left(\tfrac{1}{3}A\right)^k = Q\begin{pmatrix} 0 & & \\ & \frac{2}{3} & \\ & & 1 \end{pmatrix}^k Q^T \xrightarrow[k \to \infty]{} Q\begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} Q^T$$

$$\therefore\ u_\infty = Q\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} Q^T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = Q\begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}\begin{pmatrix} \frac{2}{\sqrt{6}} \\ 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= Q\begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}.$$