Your PRINTED name is:__________________________

Please circle your recitation:

1  T 9  2-132  Kestutis Cesnavicius  2-089  2-1195  kestutis
2  T 10  2-132  Niels Moeller  2-588  3-4110  moller
3  T 10  2-146  Kestutis Cesnavicius  2-089  2-1195  kestutis
4  T 11  2-132  Niels Moeller  2-588  3-4110  moller
5  T 12  2-132  Yan Zhang  2-487  3-4083  yanzhang
6  T 1  2-132  Taedong Yun  2-342  3-7578  tedyun
1 (13 pts.)

Suppose the matrix $A$ is the product

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) (3 pts.) What is the rank of $A$?

(b) (5 pts.) Give a basis for the nullspace of $A$.

(c) (5 pts.) For what values of $t$ (if any) are there solutions to $Ax = \begin{pmatrix} 1 \\ 1 \\ t \end{pmatrix}$?
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2 (12 pts.)

Let \( A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \).

(a) (3 pts.) Find a basis for the column space of \( A \).

(b) (3 pts.) Find a basis for the column space of \( \Sigma \) where \( A = U\Sigma V^T \) is the SVD of \( A \).

(c) (3 pts.) Find a basis for the column space of the matrix exponential \( e^A \).

(d) (3 pts.) Find a non-zero constant solution (meaning no dependence on \( t \)) to \( \frac{d}{dt}u(t) = Au(t) \).
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3 (12 pts.)

(a) (3 pts.) Give an example of a nondiagonalizable matrix $A$ which satisfies $\det(A - tI) = (4 - t)^4$

(b) (3 pts.) Give an example of two different matrices that are similar and both satisfy $\det(A - tI) = (1 - t)(2 - t)(3 - t)(4 - t)$.

(c) (3 pts.) Give an example if possible of two matrices that are not similar that satisfy $\det(A - tI) = (1 - t)(2 - t)(3 - t)(4 - t)$. 
(d) (3 pts.) Give an example of two different 4 by 4 matrices that have singular values 4, 3, 2, 1.
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4 (16 pts.)

The matrix $G = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$.

(a) (3 pts.) This matrix has two eigenvalues $\lambda = 2$, and one eigenvalue $\lambda = -2$. Given that, find the fourth eigenvalue.

(b) (5 pts.) Find a real eigenvector and show that it is indeed an eigenvector.
(Problem 4 continued.) The matrix $G = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$.

(c) (4 pts.) Is $G$ a Hermitian matrix? Why or why not. (Remember Hermitian means that $H_{jk} = \bar{H}_{kj}$ where the bar indicates complex conjugate.)

(d) (4 pts.) Give an example of a real non-diagonal matrix $X$ for which $G^H X G$ is Hermitian.
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5 (16 pts.)

The following operators apply to differentiable functions $f(x)$ transforming them to another function $g(x)$. For each one state clearly whether it is linear or not (explanations not needed). (2 pts each problem)

(a) $g(x) = \frac{d}{dx} f(x)$

(b) $g(x) = \frac{d}{dx} f(x) + 2$

(c) $g(x) = \frac{d}{dx} f(2x)$

(d) $g(x) = f(x + 2)$

(e) $g(x) = f(x)^2$

(f) $g(x) = f(x^2)$

(g) $g(x) = 0$

(h) $g(x) = f(x) + f(2)$
Let \( A = I_3 - cE_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - c \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \).

(a) (4 pts.) There are two values of \( c \) that make \( A \) a projection matrix. Find them by guessing, calculating, or understanding projection matrices. Check that \( A \) is a projection matrix for these two \( c \).

(b) (4 pts.) There are two values of \( c \) that make \( A \) an orthogonal matrix. Find them and check that \( A \) is orthogonal for these two \( c \).

(c) (4 pts.) For which values of \( c \), if any, is \( A \) diagonalizable?
(Problem 6 Continued) Let \( A = I_3 - cE_3 \) = 
\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
- c 
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}.
\]

(d) (4 pts.) Find the eigenvalues of \( A^{-1} \) (if it exists) in terms of \( c \). (Hint: find the eigenvalues of \( E_3 \) first.)

(e) (4 pts.) For which values of \( c \), if any, is \( A \) positive definite?
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7 (11 pts.)

The general equation of a circle in the plane has the form \( x^2 + y^2 + Cx + Dy + E = 0 \). Suppose you are trying to fit \( n \geq 3 \) distinct points \( p_i = (x_i, y_i), i = 1, \ldots, n \) to obtain a “best” least squares circle, it is reasonable to write a generally unsolvable equation

\[
A \begin{pmatrix} C \\ D \\ E \end{pmatrix} = b
\]

for the coefficients \( C, D, \) and \( E \).

(a) (7 pts.) Describe \( A \) and \( b \) clearly, indicating the number of rows and columns of \( A \) and the number of elements in \( b \).

(b) (4 pts.) When \( n = 3 \) it is possible to describe when the equation is and is not solvable. You can use your geometric intuition or a determinant area formula to describe the condition on the points \( p_1, p_2, p_3 \) that makes \( A \) singular. Give a simple geometrical description of this condition. (We are looking for a specific word – so only a short answer will be accepted.)
Linear Algebra is really really useful. Hope you enjoyed the class and find an opportunity to use the ideas you learned in new situations. Thanks for taking the class, have a great holiday, and wishing you all a happy 2012!