Solution Set 6, 18.06 Fall ’12

1. Do problem 9 from 4.4
   (a) We write $q_1$ and $q_2$ as the columns of a matrix $Q$:
   
   \[
   Q = \begin{pmatrix}
   .8 & .6 \\
   -.6 & .8 \\
   0 & 0
   \end{pmatrix}.
   \]

   We compute
   
   \[
   P = QQ^T = \begin{pmatrix}
   1 & 0 & 0 \\
   0 & 1 & 0 \\
   0 & 0 & 0
   \end{pmatrix},
   \]

   Indeed it is the case that $P^2 = P$.

   (b) $(QQ^T)^2 = QQ^TQQ^T = QIQ^T = QQ^T$

2. Do problem 23 from 4.4
   We take $q_1 = (1,0,0)$. We will find $q_2$ (up to scaling) by subtracting off the projection onto $q_1$.
   
   \[
   q_2' = (2,0,3) - (2,0,3) \cdot q_1 = (0,0,3).
   \]

   We now scale to find $q_2$.
   
   \[
   q_2 = q_2'/||q_2'|| = (0,0,1).
   \]

   We find $q_3$ (up to scaling) by subtracting off the projection onto $q_1$ and $q_2$.
   
   \[
   q_3' = (4,5,6) - (4,5,6) \cdot q_1 - (4,5,6) \cdot q_2 = (0,5,0).
   \]

   We now scale to find $q_3$.
   
   \[
   q_3 = q_3'/||q_3'|| = (0,1,0).
   \]

   This gives $A = QR$ where
   
   \[
   Q = \begin{pmatrix}
   1 & 0 & 0 \\
   0 & 0 & 1 \\
   0 & 1 & 0
   \end{pmatrix},
   \]
   
   \[
   R = \begin{pmatrix}
   1 & 2 & 4 \\
   0 & 3 & 6 \\
   0 & 0 & 5
   \end{pmatrix}.
   \]
3. Do problem 31 from 4.4

Each column has norm 2 so we should take \( c = 1/2 \) \((-1/2\) works as well).

To project \( b \) onto the first column we simply take

\[
\frac{1}{2}(1, -1, -1, -1) \cdot b = -1.
\]

Then the projection is just \(-\frac{1}{2}(1, -1, -1, -1)\).

To project onto the plane that is spanned by the first two columns, we add the projections on to both of them:

\[
\frac{1}{2}((1, -1, -1, -1) \cdot b) \frac{1}{2}(1, -1, -1, -1) + \frac{1}{2}(-1, 1, -1, -1) \cdot b (-1, 1, -1, -1) = \\
\frac{1}{2}((-1, 1, 1, 1) + (1, -1, 1, 1)) = (0, 0, 1, 1).
\]

4. Do problem 3 from 8.5

The zero vector is orthogonal and has length 0.

Alternatively, \((1, -2, 0, 0, 0 \ldots)\) is orthogonal and has length \(\sqrt{5}\).

5. We begin by writing their derivatives:

\[
0, -\sin(x), \cos(x), -2\sin(2x), 2\cos(2x).
\]

Then the corresponding differentiation matrix is:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 \\
0 & 0 & 0 & 2 & 0
\end{pmatrix}.
\]

6. Matlab problem; see Matlab solutions.

7. Do problem 3 from 5.1

(a) False: consider the two by two case: \( \det(I + I) = 4 \) but \( 1 + \det(I) = 2 \).

(b) True: \( |ABC| = |AB| \cdot |C| = |A| \cdot |B| \cdot |C| \).
(c) False: consider the two by two case: \( \det(4I) = 16 \) but \( 4\det(I) = 4 \).

(d) False: Take \( A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \) and \( B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \). Then \( AB - BA = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \) which has determinant \(-1\).

8. Do problem 12 from 5.1

For two by two matrices \( A \), we have

\[ |cA| = c^2|A|. \]

So the actual determinant is

\[ (ad - bc)/(ad - bc)^2 = 1/(ad - bc). \]

9. Do problem 25 from 5.1

Notice that two times the first row is the second row, so the rows are not independent so \( A \) is not invertible and so it has determinant 0.

10. We may perform the first step of the elimination to obtain some matrix \( B \) whose determinant is the same as that of \( A \), whose first column is a 1 or a \(-1\) followed by zeroes, and for which the \( 5 \times 5 \) matrix \( B' \) obtained by removing the first row and first column has entries that are all 0, 2, \(-2\).

Now, we consider the cofactor expansion along the first row.

\[
\det(A) = A_{1,1}C_{1,1} + A_{1,2}C_{1,2} + \cdots + A_{1,6}C_{1,6},
\]

where \( C_{i,j} \) is the cofactor obtained by removing the \( i \)th row and the \( j \)th column.

Then all of the terms besides the first one are zero, since by removing the first row and any column that is not the first we obtain a matrix for which the first column is all zeroes and so has determinant zero.

The first term is divisible by 32. To see this, note that \( A_{1,1}C_{1,1} \) is plus or minus 1 times the determinant of \( B' \).

Then we may factor a 2 out of \( B' \) to get that \( |B'| = 2^5|B'/2| = 32|B'/2| \). But every entry of \( B'/2 \) is an integer, and so \( |B'/2| \) is an integer (to see this, look at the permutations definition of determinant). Then 32 times an integer is divisible by 32.