1. (a) Linear. $XB$ is linear since by definition of matrix multiplication each entry of $XB$ is just a linear combination of the entries of $X$. Similarly, $AX$ is linear. Since a composition of linear transformations is linear, we also have $AXB$ is linear.

(b) Not linear. Consider any non-zero $X$. Then $(2X)^TA(2X) = 4X^TAX \neq 2X^TAX$.

(c) Linear. $AX$ and $XB$ are linear as before, and the sum of linear transformations is linear.

(d) Linear. The trace is just a linear combination of the entries of $X$.

(e) Not linear. Consider $X = I$, the 2 by 2 identity matrix. Then $\det(2I) = 4 \neq 2 \det(I)$.

2. Yes, it is linear.

We have the transformation $T(f(x)) = g(x) = f(x^2 + x)$. This is just saying that our transformation $T$ replaces each $x$ with $x^2 + x$. For linearity, we need to check that $cf(x)$ goes to $cg(x)$ and that $f_1(x) + f_2(x)$ goes to $g_1(x) + g_2(x)$. Clearly,

$$T(cf(x)) = cf(x^2 + x) = cg(x)$$

and

$$T(f_1(x) + f_2(x)) = f_1(x^2 + x) + f_2(x^2 + x) = g_1(x) + g_2(x),$$

thus this transformation is linear.

3. Problem 37 from 7.2.

We first find the result of the proposed transformation on each of the input basis “vectors” $v_1, v_2, v_3, v_4$. These can be written as linear combinations of the same basis “vectors”.

For example,

$$T(v_1) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} = av_1 + cv_3.$$

Similarly,

$$T(v_2) = av_2 + cv_4, \quad T(v_3) = bv_1 + dv_3, \quad T(v_4) = bv_2 + dv_4.$$

Noting that the transformation of the basis “vector” $v_i$ gives us the $i$th column of $A$, we conclude

$$A = \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}.$$
4. Problem 35 from 7.2.

The Haar wavelet basis for \( R^8 \) is

\[
\begin{align*}
    w_1 &= [1, 1, 1, 1, 1, 1, 1, 1]^T \\
    w_2 &= [1, 1, 1, 1, -1, -1, -1, -1]^T \\
    w_3 &= [1, 1, -1, -1, 0, 0, 0, 0]^T \\
    w_4 &= [0, 0, 0, 0, 1, 1, -1, -1]^T \\
    w_5 &= [1, -1, 0, 0, 0, 0, 0, 0]^T \\
    w_6 &= [0, 0, 1, -1, 0, 0, 0, 0]^T \\
    w_7 &= [0, 0, 0, 0, 1, -1, 0, 0]^T \\
    w_8 &= [0, 0, 0, 0, 0, 0, 1, -1]^T.
\end{align*}
\]

Note that these vectors form an orthogonal basis for \( R^8 \).

5. Problem 5 from 7.2.

\( T \) is a linear transformation from the three-dimensional space \( V \) to the three-dimensional space \( W \). \( T(v_i) \) is a combination \( a_{1i}w_1 + a_{2i}w_2 + a_{3i}w_3 \) of the output basis for \( W \). The \( a \)'s then form the \( i \)th column of the matrix \( A \). For example,

\[
    T(v_1) = 0w_1 + 1w_2 + 0w_3
\]

gives the first column: \([0, 1, 0]^T\). Repeating this we find,

\[
    T(v_2) = 1w_1 + 0w_2 + 1w_3,
\]
\[
    T(v_3) = 1w_1 + 0w_2 + 1w_3.
\]

Therefore,

\[
    A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},
\]

and

\[
    A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.
\]

This is equivalent to the fact that \( T(v_1 + v_2 + v_3) = 2w_1 + w_2 + 2w_3 \), which is demonstrable by virtue of the linearity of \( T \):

\[
    T(v_1 + v_2 + v_3) = T(v_1) + T(v_2) + T(v_3) = (w_2) + (w_1 + w_3) + (w_1 + w_3) = 2w_1 + w_2 + 2w_3.
\]
6. Problem 23 from 7.2.

We require that the matrix \( M = \begin{bmatrix} m_1 & m_4 & m_7 \\ m_2 & m_5 & m_8 \\ m_3 & m_6 & m_9 \end{bmatrix} \) is invertible, namely \( \det(M) \neq 0 \).

Note: the matrix \( M \) represents a change of basis matrix that takes parabolas in the proposed basis \( v_1, v_2, v_3 \) to a different (obviously complete) basis for parabolas \( w_1 = 1, w_2 = x, w_3 = x^2 \). Thus to be able to represent all parabolas from this complete basis \( (w_1, w_2, w_3) \) in the proposed basis \( (v_1, v_2, v_3) \), we must require that \( M^{-1} \) exists.

7. Problem 8 from 9.3.

To find \( |\lambda|_{\text{max}} \) we need to find the eigenvalues of the iteration matrix \( B = S^{-1}T \).

For **Jacobi**, \( S = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \) and \( T = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \).

So we have \( B = S^{-1}T = \begin{bmatrix} 1/a & 0 \\ 0 & 1/d \end{bmatrix} \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} = \begin{bmatrix} 0 & b/a \\ c/d & 0 \end{bmatrix} \).

The characteristic polynomial of \( B \) is \( \lambda^2 - bc/ad = 0 \) which gives \( \lambda = \pm(bc/ad)^{1/2} \).

So \( |\lambda| = |(bc/ad)^{1/2}| = |bc/ad|^{1/2} \). Therefore \( |\lambda|_{\text{max}} = |bc/ad|^{1/2} \).

For **Gauss-Seidel**, \( S = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \) and \( T = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \).

So we have \( B = S^{-1}T = \begin{bmatrix} 1/a & 0 \\ -c/ad & 1/d \end{bmatrix} \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & b/a \\ 0 & -bc/ad \end{bmatrix} \).

\( B \) is upper triangular so we can read the eigenvalues off the diagonal: \( \lambda = 0, -bc/ad \).

So \( |\lambda| = 0, |bc/ad| \). Therefore \( |\lambda|_{\text{max}} = |bc/ad| \).