18.06 Set 3 Solutions

1.

\[ A \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \]
\[ B \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]
\[ C \rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

2.

\[ A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix} \]
\[ B = \begin{bmatrix} 3 & 9 & -4.5 \\ 1 & 3 & -1.5 \\ 2 & 6 & -3 \end{bmatrix} \]
\[ M = \begin{bmatrix} a & b \\ c & bc/a \end{bmatrix} \]

3. Define \( e_i \) to be the \( i \)-th column of \( I \), which is \( n \times n \). The nullspace matrix of \( A \) has columns \( e_i - e_{i+n} \) for \( i = 1, 2, \ldots, n \). The nullspace matrix of \( B \) is the same. The nullspace matrix of \( C \) has columns \( e_i - e_{i+n} \) for \( i = 1, 2, \ldots, n \) in addition to columns \( e_{i+n} - e_{i+2n} \) for \( i = 1, \ldots, n \), a total of \( 2n \) columns.

4.

\[ A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \]
\[ b = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \]

5. The rref of \( A \) is

\[ \begin{bmatrix} 1 & 0 & -20/3 \\ 0 & 1 & 5/3 \\ 0 & 0 & 0 \end{bmatrix} \]
therefore the rank is 2. The rref of $A^t$ is
\[
\begin{bmatrix}
1 & 0 & -5 \\
0 & 1 & 2 \\
0 & 0 & 0 \\
\end{bmatrix},
\]
therefore the rank is 2. Partially reducing $A$ with $q$ we get
\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & q - 2 \\
\end{bmatrix},
\]
so if $q \neq 2$, the matrix has rank 3, and otherwise it has rank 2. Partially reducing $A^t$ with $q$ we get
\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & q - 2 \\
\end{bmatrix},
\]
which is the same situation.

6. a. $(1, 1, 1, 1)^t$, b. It is $(1, -1, 0, 0)^t$, $(0, 1, -1, 0)^t$, $(0, 0, 1, -1)^t$. c. $(0, 0, -1, 1)^t$, $(1, -1, -1, 0)^t$. d. The column space has basis $(1, 0, 0, 0)^t$, $(0, 1, 0, 0)^t$, $(0, 0, 1, 0)^t$, $(0, 0, 0, 1)^t$, the basis of the nullspace is the empty set.

7. Let
\[
A_1 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix},
\]
\[
A_2 = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{bmatrix},
\]
\[
A_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
\end{bmatrix},
\]
\[
A_4 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
\end{bmatrix},
\]
\[
A_5 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
\end{bmatrix},
\]
\[
A_6 = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\end{bmatrix}
\]
The trick is $A_1 + A_5 + A_6 = A_2 + A_3 + A_4$ which is the matrix of all 1’s, so $A_1 = A_2 + A_3 + A_4 - A_5 - A_6$. Now let $c \in \mathbb{R}^6$ such that $c_2 A_2 + c_3 A_3 + c_4 A_4 + c_5 A_5 + c_6 A_6 = 0$ (there is no $c_1$, $c$ is indexed from 2 to 6). $c_2$ must be 0 since $A_2$ has a 1 in the lower right hand corner and no other $A_i$ has that for $i \neq 1$. Likewise $c_3$ must be 0 since $A_3$ has a 1 in the upper left hand corner and no other $A_i$ has that for $i \neq 1$. $c_4 = 0$ since $A_4$ has a 1 in the dead center. It is obvious that $A_5$ and $A_6$ are linearly independent from each other, hence $c_5 = c_6 = 0$. So $c = \vec{0}$ and the matrices are linearly independent.

8. a. $y = c$, a constant. b. $y = 3x$. c. $y = 3x + c$.

9. Integer matrices: rank zero 0%,