Solutions to Alex's review questions #3

ad problem 1: <u>The first matrix:</u>

Let A be the first matrix, $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

(a) The eigenvalues are the roots of the characteristic polynomial

$$\det(A - \lambda I_2) = \det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = \lambda^2 - 4\lambda + 3.$$

These are 1 and 3.

The eigenvectors for 1 are the elements of the nullspace of $A - 1I_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. They form a 1-dimensional vector space spanned by $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

The eigenvectors for 3 are the elements of the nullspace of $A - 3I_2 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$. They form a 1-dimensional vector space spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(b) The matrix A is diagonalizable. Thus, p. 312 of the book tells us how to solve the equation $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$: Its solutions are the linear combinations of functions of the form $e^{\lambda t}\mathbf{x}$ (as a function in t) with λ being an eigenvalue of A and $\mathbf{x}$ being an eigenvector corresponding to λ. Of course, we do not need to go through all eigenvectors of A; it is enough to use a basis of the eigenspace corresponding to each eigenvalue.

So the solutions in our case have the form $\mathbf{u}(t) = ae^{1t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + be^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with $a, b \in \mathbb{C}$ (because $\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$ is a basis of the eigenspace to eigenvalue 1, and $\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$ is a basis of the eigenspace to eigenvalue 3). In other words, they have the form $\mathbf{u}(t) = \begin{pmatrix} ae^t + be^{3t} \\ -ae^t + be^{3t} \end{pmatrix}$ with $a, b \in \mathbb{C}$.

(c) We know that the solution has the form $\mathbf{u}(t) = \begin{pmatrix} ae^t + be^{3t} \\ -ae^t + be^{3t} \end{pmatrix}$ for some $a, b \in \mathbb{C}$. We now need to find the a, b satisfying $\mathbf{u}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Since $\mathbf{u}(0) = \begin{pmatrix} ae^0 + be^{3\cdot 0} \\ -ae^0 + be^{3\cdot 0} \end{pmatrix} = \begin{pmatrix} a+b \\ -a+b \end{pmatrix}$, this becomes $\begin{pmatrix} a+b \\ -a+b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. This is a system of linear equations in a, b, and its only solution is $(a, b) = (0, 1)$. Thus, $a = 0$ and $b = 1$, so that $\mathbf{u}(t) = \begin{pmatrix} ae^t + be^{3t} \\ -ae^t + be^{3t} \end{pmatrix} = \begin{pmatrix} 0e^t + 1e^{3t} \\ -0e^t + 1e^{3t} \end{pmatrix} = \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}$.

(e) We do **(e)** before doing **(d)** because this is the easier way.

We have $A = S\Lambda S^{-1}$ where $S = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and $\Lambda = \operatorname{diag}(1,3) = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$. (See page 298 for how we obtained S and Λ. Notice that $\operatorname{diag}(a_1, a_2, \ldots, a_k)$ always means the diagonal $k \times k$-matrix whose diagonal entries, from upper-left to lower-right, are $a_1, a_2, \ldots, a_k$.) So Λ is a diagonal matrix which is similar to A.

(d) We have $A = S\Lambda S^{-1}$, so $e^{At} = e^{S\Lambda S^{-1}t} = e^{S(\Lambda t)S^{-1}} = Se^{\Lambda t}S^{-1}$. But $\Lambda = \operatorname{diag}(1,3)$, so $\Lambda t = \operatorname{diag}(1t, 3t)$ and thus $e^{\Lambda t} = \operatorname{diag}(e^{1t}, e^{3t}) = \operatorname{diag}(e^t, e^{3t})$. Thus,

$$e^{At} = \underbrace{S}_{=\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}} \underbrace{e^{\Lambda t}}_{=\operatorname{diag}(e^t, e^{3t}) = \begin{pmatrix} e^t & 0 \\ 0 & e^{3t} \end{pmatrix}} \underbrace{S^{-1}}_{=\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1}} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^t & \frac{1}{2}e^{3t} - \frac{1}{2}e^t \\ \frac{1}{2}e^{3t} - \frac{1}{2}e^t & \frac{1}{2}e^{3t} + \frac{1}{2}e^t \end{pmatrix}.$$

(f) The matrix A can be factored like this, because A is symmetric and every symmetric matrix can be factored like this (see page 330).

[*Remark:* Actually, a factorization of A as $Q\Lambda Q^T$ could be obtained from our $A = S\Lambda S^{-1}$ factorization by setting $Q = \frac{1}{\sqrt{2}}S$. This is not a general method to obtain $Q\Lambda Q^T$ factorizations of symmetric matrices, because the eigenvectors we took for columns in S might not become orthonormal after just a simple scaling. But it is never far from S to Q; all that one needs to do is perform Gram-Schmidt orthonormalization in each eigenspace.]

<u>The second matrix:</u> with thanks to whoever invented copy & paste.

Let A be the second matrix, $\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$.

(a) The eigenvalues are the roots of the characteristic polynomial

$$\det(A - \lambda I_2) = \det \begin{pmatrix} -1-\lambda & 0 \\ 1 & 1-\lambda \end{pmatrix} = \lambda^2 - 1.$$

These are 1 and -1.

The eigenvectors for 1 are the elements of the nullspace of $A - 1I_2 = \begin{pmatrix} -2 & 0 \\ 1 & 0 \end{pmatrix}$. They form a 1-dimensional vector space spanned by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The eigenvectors for -1 are the elements of the nullspace of $A - (-1)I_2 = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$. They form a 1-dimensional vector space spanned by $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

(b) The matrix A is diagonalizable. Thus, p. 312 of the book tells us how to solve the equation $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$: Its solutions are the linear combinations of functions of the form $e^{\lambda t}\mathbf{x}$ (as a function in t) with λ being an eigenvalue of A and $\mathbf{x}$ being an eigenvector corresponding to λ. Of course, we do not need to go through all eigenvectors of A; it is enough to use a basis of the eigenspace corresponding to each eigenvalue.

So the solutions in our case have the form $\mathbf{u}(t) = ae^{1t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + be^{(-1)t}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ with $a, b \in \mathbb{C}$ (because $\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$ is a basis of the eigenspace to eigenvalue 1, and $\left(\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right)$ is a basis of the eigenspace to eigenvalue -1). In other words, they have the form $\mathbf{u}(t) = \begin{pmatrix} 2be^{-t} \\ ae^t - be^{-t} \end{pmatrix}$ with $a, b \in \mathbb{C}$.

(c) We know that the solution has the form $\mathbf{u}(t) = \begin{pmatrix} 2be^{-t} \\ ae^t - be^{-t} \end{pmatrix}$ for some $a, b \in \mathbb{C}$. We now need to find the a, b satisfying $\mathbf{u}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Since $\mathbf{u}(0) = \begin{pmatrix} 2be^{-0} \\ ae^0 - be^{-0} \end{pmatrix} = \begin{pmatrix} 2b \\ a - b \end{pmatrix}$, this becomes $\begin{pmatrix} 2b \\ a - b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. This is a system of linear equations in a, b, and its only solution is $(a, b) = \left(\frac{3}{2}, \frac{1}{2}\right)$. Thus, $a = \frac{3}{2}$ and $b = \frac{1}{2}$, so that $\mathbf{u}(t) = \begin{pmatrix} 2be^{-t} \\ ae^t - be^{-t} \end{pmatrix} = \begin{pmatrix} 2 \cdot \frac{1}{2}e^{-t} \\ \frac{3}{2}e^t - \frac{1}{2}e^{-t} \end{pmatrix} = \begin{pmatrix} e^{-t} \\ \frac{3}{2}e^t - \frac{1}{2}e^{-t} \end{pmatrix}$.

(e) We do **(e)** before doing **(d)** because this is the easier way.

We have $A = S\Lambda S^{-1}$ where $S = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$ and $\Lambda = \operatorname{diag}(1, -1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. (See page 298 for how we obtained S and Λ.) So Λ is a diagonal matrix which is similar to A.

(d) We have $A = S\Lambda S^{-1}$, so $e^{At} = e^{S\Lambda S^{-1}t} = e^{S(\Lambda t)S^{-1}} = Se^{\Lambda t}S^{-1}$. But $\Lambda = \operatorname{diag}(1, -1)$, so $\Lambda t = \operatorname{diag}(1t, -1t)$ and thus $e^{\Lambda t} = \operatorname{diag}\left(e^{1t}, e^{-1t}\right) = \operatorname{diag}\left(e^t, e^{-t}\right)$. Thus,

$$e^{At} = \underbrace{S}_{=\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}} \underbrace{e^{\Lambda t}}_{=\operatorname{diag}\left(e^t, e^{-t}\right) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}} \underbrace{S^{-1}}_{=\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}^{-1}} = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{1}{e^t} & 0 \\ \frac{1}{2}e^t - \frac{1}{2}e^{-t} & e^t \end{pmatrix}.$$

(f) The matrix A cannot be factored like this, because A is not symmetric but every matrix of the form $Q\Lambda Q^T$ with diagonal Λ is symmetric (in fact, every such matrix satisfies $\left(Q\Lambda Q^T\right)^T = \underbrace{\left(Q^T\right)^T}_{=Q} \underbrace{\Lambda^T}_{=\Lambda \text{ (since } \Lambda \text{ is diagonal)}} Q^T = Q\Lambda Q^T$).

<u>The third matrix:</u>

Let A be the third matrix, $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$.

(a) The eigenvalues are the roots of the characteristic polynomial

$$\det\left(A - \lambda I_2\right) = \det \begin{pmatrix} 1-\lambda & 2 \\ 3 & 6-\lambda \end{pmatrix} = \lambda^2 - 7\lambda.$$

These are 7 and 0.

The eigenvectors for 7 are the elements of the nullspace of $A - 7I_2 = \begin{pmatrix} -6 & 2 \\ 3 & -1 \end{pmatrix}$. They form a 1-dimensional vector space spanned by $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

The eigenvectors for 0 are the elements of the nullspace of $A - 0I_2 = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$. They form a 1-dimensional vector space spanned by $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

(b) The matrix A is diagonalizable. Thus, p. 312 of the book tells us how to solve the equation $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$: Its solutions are the linear combinations of functions of the form $e^{\lambda t}\mathbf{x}$ (as a function in t) with λ being an eigenvalue of A and $\mathbf{x}$ being an eigenvector corresponding to λ. Of course, we do not need to go through all eigenvectors of A; it is enough to use a basis of the eigenspace corresponding to each eigenvalue.

So the solutions in our case have the form $\mathbf{u}(t) = ae^{7t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + be^{0t}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ with $a, b \in \mathbb{C}$ (because $\left(\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right)$ is a basis of the eigenspace to eigenvalue 7, and $\left(\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right)$ is a basis of the eigenspace to eigenvalue 0). In other words, they have the form $\mathbf{u}(t) = \begin{pmatrix} ae^{7t} + 2b \\ 3ae^{7t} - b \end{pmatrix}$ with $a, b \in \mathbb{C}$.

(c) We know that the solution has the form $\mathbf{u}(t) = \begin{pmatrix} ae^{7t} + 2b \\ 3ae^{7t} - b \end{pmatrix}$ for some $a, b \in \mathbb{C}$. We now need to find the a, b satisfying $\mathbf{u}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Since $\mathbf{u}(0) = \begin{pmatrix} ae^{7\cdot 0} + 2b \\ 3ae^{7\cdot 0} - b \end{pmatrix} = \begin{pmatrix} a + 2b \\ 3a - b \end{pmatrix}$, this becomes $\begin{pmatrix} a + 2b \\ 3a - b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. This is a system of linear equations in a, b, and its only solution is $(a, b) = \left(\frac{3}{7}, \frac{2}{7}\right)$. Thus,

$a = \frac{3}{7}$ and $b = \frac{2}{7}$, so that $\mathbf{u}(t) = \begin{pmatrix} \frac{3}{7}e^{7t} + 2 \cdot \frac{2}{7} \\ 3 \cdot \frac{3}{7}e^{7t} - \frac{2}{7} \end{pmatrix} = \begin{pmatrix} \frac{3}{7}e^{7t} + \frac{4}{7} \\ \frac{9}{7}e^{7t} - \frac{2}{7} \end{pmatrix}$.

(e) We do **(e)** before doing **(d)** because this is the easier way.

We have $A = S\Lambda S^{-1}$ where $S = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$ and $\Lambda = \operatorname{diag}(7, 0) = \begin{pmatrix} 7 & 0 \\ 0 & 0 \end{pmatrix}$. (See page 298 for how we obtained S and Λ.) So Λ is a diagonal matrix which is similar to A.

(d) We have $A = S\Lambda S^{-1}$, so $e^{At} = e^{S\Lambda S^{-1}t} = e^{S(\Lambda t)S^{-1}} = Se^{\Lambda t}S^{-1}$. But $\Lambda = \operatorname{diag}(7, 0)$, so $\Lambda t = \operatorname{diag}(7t, 0t)$ and thus $e^{\Lambda t} = \operatorname{diag}\left(e^{7t}, e^{0t}\right) = \operatorname{diag}\left(e^{7t}, 1\right)$. Thus,

$$e^{At} = \underbrace{S}_{=\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}} \underbrace{e^{\Lambda t}}_{=\operatorname{diag}\left(e^{7t},1\right) = \begin{pmatrix} e^{7t} & 0 \\ 0 & 1 \end{pmatrix}} \underbrace{S^{-1}}_{=\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}^{-1}} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} e^{7t} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{1}{7}e^{7t} + \frac{6}{7} & \frac{2}{7}e^{7t} - \frac{2}{7} \\ \frac{3}{7}e^{7t} - \frac{3}{7} & \frac{6}{7}e^{7t} + \frac{1}{7} \end{pmatrix}.$$

(f) The matrix A cannot be factored like this, because A is not symmetric but every matrix of the form $Q\Lambda Q^T$ with diagonal Λ is symmetric.

ad problem 2: The matrix A is symmetric, and therefore so is $-A$. Hence, speaking of positive definiteness for A and $-A$ makes sense.

(a) Let us recall that a symmetric $n \times n$-matrix B is positive definite if and only if all n upper-left determinants of B are positive.[1] Applying this to our 3×3-matrix A, we see that A is positive definite if and only if its upper-left determinants

$$\det(2) = 2,$$

$$\det \begin{pmatrix} 2 & 1 \\ 1 & a \end{pmatrix} = 2a - 1, \qquad \text{and}$$

$$\det \begin{pmatrix} 2 & 1 & 1 \\ 1 & a & 2 \\ 1 & 2 & a \end{pmatrix} = 2a^2 - 2a - 4 = 2(a+1)(a-2)$$

[1] Recall that the k-th *upper-left determinant* of B (for $1 \leq k \leq n$) is the determinant of the matrix obtained by taking only the first k rows and the first k columns of B. For instance, the 2-nd upper-left determinant of $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 6 & 4 \\ 4 & 4 & 3 \end{pmatrix}$ is $\det \begin{pmatrix} 1 & 2 \\ 2 & 6 \end{pmatrix} = 2$. The n-th upper-left determinant of B is $\det B$, and the 1-st upper-left determinant of B is the first entry of the first row of B.

are positive. This boils down to the inequalities $2a - 1 > 0$ and $(a+1)(a-2) > 0$ (because 2 is always positive). These two inequalities hold if and only if we have $a > 2$. Hence, the answer is "for those a which satisfy $a > 2$".

(b) Again, let us recall that a symmetric $n \times n$-matrix B is positive definite if and only if all n upper-left determinants of B are positive. Applying this to our 3×3-matrix $-A$, we see that $-A$ is not positive definiten, since its very first upper-left determinant is $\det(-2) = -2 \leq 0$. So the answer is "never".

(c) The matrix A is singular if and only if $\det A = 0$. Since

$$\det A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & a & 2 \\ 1 & 2 & a \end{pmatrix} = 2a^2 - 2a - 4 = 2(a+1)(a-2),$$

this happens precisely when $(a+1)(a-2) = 0$, that is, when $a = 2$ or $a = -1$. Thus, the answer is "for those a which satisfy $a = 2$ or $a = -1$".

ad problem 3: (a) We follow the method on page 330.

The matrix A has eigenvalues 1 and 9. Its eigenvectors for eigenvalue 1 form a 1-dimensional space with basis $\left(\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)$; thus, an **orthonormal** basis for this space is $\left(\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}\right)$. Its eigenvectors for eigenvalue 9 form a 1-dimensional space with basis $\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$; thus, an **orthonormal** basis for this space is $\left(\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}\right)$. (Generally, we need Gram-Schmidt orthonormalization to obtain an orthonormal basis of a vector space. But in the case of 1-dimensional spaces, it is just enough to take a nonzero vector and scale it so that it has length 1.)

Thus, for the $A = Q\Lambda Q^T$ factorization, we take $\Lambda = \operatorname{diag}(1,9) = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$ (the diagonal matrix whose diagonal entries are the eigenvalues of A) and $Q = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ (the orthogonal matrix whose columns are the eigenvectors of A, in the appropriate order[2]).

[2] "Appropriate order" means that if λ_k is the k-th entry on the diagonal of Λ, then the k-th column of Q has to be an eigenvector of A for eigenvalue λ_k. So we had to take $Q =$

[There is some freedom in the choice of Q, since there are two ways to scale a vector to length 1. For instance, we could have taken $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$ instead of $\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$. We could have also had $\Lambda = \operatorname{diag}(9,1)$ instead of $\Lambda = \operatorname{diag}(1,9)$, but then the columns of Q would have to be ordered accordingly.]

(b) As we know, in order to find an LDL^T decomposition of a symmetric matrix A, we form the LDU decomposition (with L and U both having 1's on their diagonals) and just notice that $U = L^T$. And in order to find the LDU decomposition, we compute the LU decomposition (using Gaussian elimination) and then change the pivots on the diagonal of U into 1's by appropriately rescaling the rows of U (obtaining the D factor in the process). The LDU decomposition for A is $A = LDU$ with $L = \begin{pmatrix} 1 & 0 \\ \frac{4}{5} & 1 \end{pmatrix}$, $D = \begin{pmatrix} 5 & 0 \\ 0 & \frac{9}{5} \end{pmatrix}$ and $U = \begin{pmatrix} 1 & \frac{4}{5} \\ 0 & 1 \end{pmatrix} = L^T$. Thus, this is also the LDL^T decomposition of A.

(c) The pivots of A are the entries of the matrix D we have just obtained. They are 5 and $\frac{9}{5}$.

The eigenvalues of A are 1 and 9.

(d) We know Q and Λ already, and from $\Lambda = \operatorname{diag}(1,9)$ we obtain $\sqrt{\Lambda} = \operatorname{diag}\left(\sqrt{1}, \sqrt{9}\right) = \operatorname{diag}(1,3) = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$. The rest is just computation:

$$Q\sqrt{\Lambda}Q^T = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}^T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

(e) One way to do this is just take $C = Q\sqrt{\Lambda}Q^T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. This works

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$\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ here; $Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ would not have worked!

because

$$C^TC = \left(Q\sqrt{\Lambda}Q^T\right)^T \left(Q\sqrt{\Lambda}Q^T\right) = \underbrace{\left(Q^T\right)^T}_{=Q} \underbrace{\left(\sqrt{\Lambda}\right)^T}_{\substack{=\sqrt{\Lambda} \\ \text{(since } \sqrt{\Lambda} \text{ is diagonal)}}} \underbrace{Q^TQ}_{\substack{=I \\ \text{(since } Q \text{ is orthogonal)}}} \sqrt{\Lambda}Q^T$$

$$= Q \underbrace{\sqrt{\Lambda}I\sqrt{\Lambda}}_{=\sqrt{\Lambda}\sqrt{\Lambda}=\Lambda} Q^T = Q\Lambda Q^T = A.$$

Other options to find C satisfying $A = C^TC$ would be to take $C = Q\sqrt{\Lambda}$ or to take $C = L\sqrt{D}$. These would all give different C's, which should not be surprising (there are infinitely many C's which do the trick).

ad problem 4: (a) The characteristic polynomials of these five matrices are (in order)

$$\det\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \lambda I_2\right) = \det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = \lambda^2 - 4\lambda + 3;$$

$$\det\left(\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} - \lambda I_2\right) = \det\begin{pmatrix} -1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} = \lambda^2 - 1;$$

$$\det\left(\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} - \lambda I_2\right) = \det\begin{pmatrix} -1-\lambda & 1 \\ 0 & -1-\lambda \end{pmatrix} = \lambda^2 + 2\lambda + 1;$$

$$\det\left(\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} - \lambda I_2\right) = \det\begin{pmatrix} 1-\lambda & 2 \\ 3 & 6-\lambda \end{pmatrix} = \lambda^2 - 7\lambda;$$

$$\det\left(\begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix} - \lambda I_2\right) = \det\begin{pmatrix} 2-\lambda & 1 \\ -3 & -2-\lambda \end{pmatrix} = \lambda^2 - 1.$$

Just looking at these polynomials, we see that the only two of these matrices which have any chance of being similar are the second and the fifth one (because two similar matrices must have the same characteristic polynomial[3]). Let us check if they are.

Indeed, the matrices $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}$ **are** similar. One way to see this is by noticing that they have the same characteristic polynomial **and** are diagonalizable (because their characteristic polynomial $\lambda^2 - 1$ has no double roots). But let us show another way, which solves part **(b)** at the same time. We set $A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}$, and we are looking for an invertible 2×2-matrix M such that $B = M^{-1}AM$. Set $M = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. The equation

[3]The converse is not true: Two matrices having the same characteristic polynomial may and may not be similar. They have to be similar if they are both diagonalizable, though.

$B = M^{-1}AM$ can be rewritten equivalently:

$$\begin{aligned}
& B = M^{-1}AM \\
\iff & MB = AM \\
\iff & \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \\
\iff & \begin{pmatrix} 2x - 3y & x - 2y \\ 2z - 3w & z - 2w \end{pmatrix} = \begin{pmatrix} z - x & w - y \\ z & w \end{pmatrix} \\
\iff & \begin{cases} 2x - 3y = z - x \\ x - 2y = w - y \\ 2z - 3w = z \\ z - 2w = w \end{cases} .
\end{aligned}$$

This is a system of four linear equations on the four unknowns x, y, z, w. We can solve it in the usual ways, and the result is that the vector $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ must be a linear combination of $\begin{pmatrix} 1 \\ 0 \\ 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$. Now, we need to find such a linear combination $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ for which the matrix $M = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ is invertible. It is easy to see that the first vector, $\begin{pmatrix} 1 \\ 0 \\ 3 \\ 1 \end{pmatrix}$, does the trick: $M = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ is invertible. So we have $B = M^{-1}AM$ for the invertible matrix $M = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$. (Of course, you could have chosen another linear combination, as long as the resulting M would be solvable – which it is almost always. The M is far from being unique.)

(b) For $A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}$, we have already found an M satisfying $B = M^{-1}AM$ in the solution to part **(a)** (it was $M = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$, but you may have found a different M).

For $A = \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ (these are the same A and B as before, but switched), it is easy to find an M satisfying $B = M^{-1}AM$, just by inverting the M found in the previous case. (So from our $M = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$, we now obtain $M = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$. Again, you might have different results.)

(c) A matrix whose characteristic polynomial has no double roots can always be diagonalized (because such a matrix has no repeated eigenvalues; see Remark 1 on page 299). It is easy to check that our first, our second, our fourth, and our fifth matrix have this property (their characteristic polynomials $\lambda^2 - 4\lambda + 3$, $\lambda^2 - 1$, $\lambda^2 - 7\lambda$ and $\lambda^2 - 1$ have no double roots), and therefore can be diagonalized.

[To say a bit more: They can all be diagonalized **over the reals**, because their characteristic polynomials all can be factored over $\mathbb{R}$.]

On the other hand, the third matrix has characteristic polynomial $\lambda^2 + 2\lambda + 1$, which does have a double root. So it might be non-diagonalizable. To see whether it is diagonalizable or not, we compute its eigenvectors for eigenvalue -1 (its only eigenvalue, appearing twice as a root of its characteristic polynomial). The only such eigenvector (up to scaling) is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. So we have only one eigenvector (up to scaling). Thus, there exists no basis of eigenvalues for the third matrix, and thus the third matrix is non-diagonalizable.

So the answer is "the third one".

ad problem 5: (a) We follow the first algorithm in Michael's recitation #10 notes[4]. Let us first repeat the algorithm:

> We want to compute an SVD of an $m \times n$-matrix A.
>
> *Step 1:* Let $\lambda_1, \lambda_2, \ldots, \lambda_i$ be the nonzero eigenvalues of $A^T A$, and $\lambda_{i+1}, \lambda_{i+2}, \ldots, \lambda_n$ be the zero eigenvalues of $A^T A$. (If you want the diagonal entries of the Σ matrix to be ordered decreasingly, you should order them such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ here.) Choose corresponding **orthonormal** eigenvectors $v_1, v_2, \ldots, v_n$ for $A^T A$.
>
> *Step 2:* Let $\sigma_j = \sqrt{\lambda_j}$ for all j. Let $u_1, u_2, \ldots, u_i$ be given by $u_j = A v_j / \sigma_j$.
>
> *Step 3:* Let $(u_{i+1}, u_{i+2}, \ldots, u_m)$ be an **orthonormal** basis of $N(AA^T)$.
>
> *Step 4:* An SVD of A is $A = U \Sigma V^T$, where:
>
> - U is the $m \times m$ matrix with columns $u_1, u_2, \ldots, u_m$;
> - V is the $n \times n$ matrix with columns $v_1, v_2, \ldots, v_n$;
> - Σ is the $m \times n$ matrix with (j, j)-th entry σ_j and all other entries 0.
>
> Note that U and V are orthogonal matrices.

[4] http://web.mit.edu/18.06/www/Fall14/Recitation10_Michael.pdf

We now have to perform this algorithm for the three matrices. There are some choices to be done here. One choice is in which order we list the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (the only important thing is that the zero eigenvalues should come at the very end); we choose to order them decreasingly. We can also choose each of $v_1, v_2, \ldots, v_n$ in at least two different ways (because if v is a length-1 eigenvector for some eigenvalue λ, then so is $-v$); if some eigenspaces are more than 1-dimensional, then we have even more freedom there. Do not be surprised if my results differ from yours.

First matrix: Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$. Thus, $m = 2$ and $n = 2$.

We have $A^T A = \begin{pmatrix} 10 & 20 \\ 20 & 40 \end{pmatrix}$ and $AA^T = \begin{pmatrix} 5 & 15 \\ 15 & 45 \end{pmatrix}$.

Step 1: The eigenvalues of $A^T A$ are 50 and 0. Thus, we have to set $\lambda_1 = 50$ and $\lambda_2 = 0$, and $i = 1$.

For the respective orthonormal eigenvectors, we find $v_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{1}{5}\sqrt{5} \\ \frac{2}{5}\sqrt{5} \end{pmatrix}$ (a length-1 eigenvector for eigenvalue λ_1) and $v_2 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} -\frac{2}{5}\sqrt{5} \\ \frac{1}{5}\sqrt{5} \end{pmatrix}$ (a length-1 eigenvector for eigenvalue λ_2).

Step 2: We have $\sigma_1 = \sqrt{50} = 5\sqrt{2}$ and $\sigma_2 = \sqrt{0} = 0$.

We have $u_1 = Av_1/\sigma_1 = \begin{pmatrix} \frac{1}{10}\sqrt{10} \\ \frac{3}{10}\sqrt{10} \end{pmatrix}$. (There is no u_2 yet, since $i = 1$.)

Step 3: We need to find an orthonormal basis (u_2) of $N\left(AA^T\right)$.

A basis of $N\left(AA^T\right)$ is $\left(\begin{pmatrix} -3 \\ 1 \end{pmatrix}\right)$. Thus, an orthonormal basis of $N\left(AA^T\right)$ is $\left(\begin{pmatrix} \frac{-3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{pmatrix}\right)$. So we take $u_2 = \begin{pmatrix} \frac{-3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{pmatrix} = \begin{pmatrix} -\frac{3}{10}\sqrt{10} \\ \frac{1}{10}\sqrt{10} \end{pmatrix}$. (We could also have taken the negative of this vector.)

Step 4: An SVD of A is $A = U\Sigma V^T$, where

$$U = \begin{pmatrix} \frac{1}{10}\sqrt{10} & -\frac{3}{10}\sqrt{10} \\ \frac{3}{10}\sqrt{10} & \frac{1}{10}\sqrt{10} \end{pmatrix};$$

$$\Sigma = \begin{pmatrix} 5\sqrt{2} & 0 \\ 0 & 0 \end{pmatrix};$$

$$V = \begin{pmatrix} \frac{1}{5}\sqrt{5} & -\frac{2}{5}\sqrt{5} \\ \frac{2}{5}\sqrt{5} & \frac{1}{5}\sqrt{5} \end{pmatrix}.$$

<u>Second matrix:</u> Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. Thus, $m = 2$ and $n = 3$.

Since $m < n$, it might be simpler to take the second algorithm suggested by Michael. But then it would be harder for me to copy&paste, so let me use the same algorithm as before.

We have $A^T A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and $AA^T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Step 1: The eigenvalues of $A^T A$ are 3, 1 and 0. Thus, we have to set $\lambda_1 = 3$, $\lambda_2 = 1$, $\lambda_3 = 0$, and $i = 2$. (We could have set $\lambda_1 = 1$ and $\lambda_2 = 3$ as well, but we must set $\lambda_3 = 0$.)

For the respective orthonormal eigenvectors, we find

- $v_1 = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} \frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{6} \\ \frac{1}{6}\sqrt{6} \end{pmatrix}$ (a length-1 eigenvector for eigenvalue λ_1),
- $v_2 = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ 0 \\ \frac{1}{2}\sqrt{2} \end{pmatrix}$ (a length-1 eigenvector for eigenvalue λ_2),
- $v_3 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} \frac{1}{3}\sqrt{3} \\ -\frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \end{pmatrix}$ (a length-1 eigenvector for eigenvalue λ_3).

Step 2: We have $\sigma_1 = \sqrt{3}$, $\sigma_2 = \sqrt{1} = 1$ and $\sigma_3 = \sqrt{0} = 0$.

We have $u_1 = Av_1/\sigma_1 = \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}$ and $u_2 = Av_2/\sigma_2 = \begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}$.

Step 3: We need to find an orthonormal basis $()$ of $N\left(AA^T\right)$. Here, $()$ means the empty list. Why the empty list? Well, the algorithm tells us to find an orthonormal basis $(u_{i+1}, u_{i+2}, \ldots, u_m)$ of $N\left(AA^T\right)$. Since $i = 2$ and $m = 3$, we must have $(u_{i+1}, u_{i+2}, \ldots, u_m) = (u_3, u_4, \ldots, u_2) = ()$. (And $N\left(AA^T\right)$ is the zero space, so the empty list **is** an orthonormal basis of it.) How do we find an empty list? Well, we found it already. :)

Step 4: An SVD of A is $A = U\Sigma V^T$, where

$$U = \begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix};$$

$$\Sigma = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix};$$

$$V = \begin{pmatrix} \frac{1}{6}\sqrt{6} & -\frac{1}{2}\sqrt{2} & \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{6} & 0 & -\frac{1}{3}\sqrt{3} \\ \frac{1}{6}\sqrt{6} & \frac{1}{2}\sqrt{2} & \frac{1}{3}\sqrt{3} \end{pmatrix}.$$

Third matrix: Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$. Thus, $m = 3$ and $n = 2$.

The short way here would be recognize this A as the transpose of the second matrix, and apply the result of pset #10 problem 1 **(a)**. But we shall follow the same algorithm as in the two previous cases.

We have $A^TA = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $AA^T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

Step 1: The eigenvalues of AA^T are 3 and 1. Thus, we have to set $\lambda_1 = 3$, $\lambda_2 = 1$, and $i = 2$. (We could have set $\lambda_1 = 1$ and $\lambda_2 = 3$ as well.)

For the respective orthonormal eigenvectors, we find

- $v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}$ (a length-1 eigenvector for eigenvalue λ_1),

- $v_2 = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}$ (a length-1 eigenvector for eigenvalue λ_2).

Step 2: We have $\sigma_1 = \sqrt{3}$ and $\sigma_2 = \sqrt{1} = 1$.

We have $u_1 = Av_1/\sigma_1 = \begin{pmatrix} \frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{6} \\ \frac{1}{6}\sqrt{6} \end{pmatrix}$ and $u_2 = Av_2/\sigma_2 = \begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ 0 \\ \frac{1}{2}\sqrt{2} \end{pmatrix}$

Step 3: We need to find an orthonormal basis (u_3) of $N\left(AA^T\right)$. We compute its single element to be $u_3 = \begin{pmatrix} \frac{1}{3}\sqrt{3} \\ -\frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \end{pmatrix}$ (or its negative, but we choose this one).

Step 4: An SVD of A is $A = U\Sigma V^T$, where

$$U = \begin{pmatrix} \frac{1}{6}\sqrt{6} & -\frac{1}{2}\sqrt{2} & \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{6} & 0 & -\frac{1}{3}\sqrt{3} \\ \frac{1}{6}\sqrt{6} & \frac{1}{2}\sqrt{2} & \frac{1}{3}\sqrt{3} \end{pmatrix};$$

$$\Sigma = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix};$$

$$V = \begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}.$$

(b) The way how to do this is explained in the very beginning of page 368 of the book (but Strang's r is our i).

First matrix: Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$. Orthonormal bases:

- for row space of A: $\left(\begin{pmatrix} \frac{1}{5}\sqrt{5} \\ \frac{2}{5}\sqrt{5} \end{pmatrix}\right)$.

- for nullspace of A: $\left(\begin{pmatrix} -\frac{2}{5}\sqrt{5} \\ \frac{1}{5}\sqrt{5} \end{pmatrix}\right)$.

- for column space of A: $\left(\begin{pmatrix} \frac{1}{10}\sqrt{10} \\ \frac{3}{10}\sqrt{10} \end{pmatrix}\right)$.
- for nullspace of A^T: $\left(\begin{pmatrix} -\frac{3}{10}\sqrt{10} \\ \frac{1}{10}\sqrt{10} \end{pmatrix}\right)$.

Second matrix: Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. Orthonormal bases:

- for row space of A: $\left(\begin{pmatrix} \frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{6} \\ \frac{1}{6}\sqrt{6} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ 0 \\ \frac{1}{2}\sqrt{2} \end{pmatrix}\right)$.
- for nullspace of A: $\left(\begin{pmatrix} \frac{1}{3}\sqrt{3} \\ -\frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \end{pmatrix}\right)$.
- for column space of A: $\left(\begin{pmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}\right)$.
- for nullspace of A^T: $()$ (an empty basis for a zero space.

Third matrix: Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$. Orthonormal bases:

- for row space of A: $\left(\begin{pmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}\right)$.
- for nullspace of A: $()$ (an empty basis for a zero space.
- for column space of A: $\left(\begin{pmatrix} \frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{6} \\ \frac{1}{6}\sqrt{6} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ 0 \\ \frac{1}{2}\sqrt{2} \end{pmatrix}\right)$.

- for nullspace of A^T: $\left(\left(\begin{array}{c}\frac{1}{3}\sqrt{3}\\ -\frac{1}{3}\sqrt{3}\\ \frac{1}{3}\sqrt{3}\end{array}\right)\right)$.

(c) We have ordered the $\lambda_1, \lambda_2, \ldots, \lambda_i$ decreasingly, so λ_1 is the biggest eigenvalue of $A^T A$, and thus σ_1 is the biggest singular value of A. Thus:

First matrix: $\sigma_1 u_1 v_1^T = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$. [This is exactly A. Surprising? No, since A is already rank-1.]

Second matrix: $\sigma_1 u_1 v_1^T = \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}$.

Third matrix: $\sigma_1 u_1 v_1^T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.