Your PRINTED name is: __________________________

Please CIRCLE your section:

Grading 1:

| R01 T 9  | E17-128 | Miriam Farber  |
| R02 T 10 | 38-166  | Sam Raskin    |
| R03 T 10 | E17-128 | Miriam Farber  |
| R04 T 11 | 38-166  | Sam Raskin    |
| R05 T 12 | E17-133 | Nate Harman   |
| R06 T 1  | E17-139 | Tanya Khovanova |
| R07 T 2  | E17-133 | Tanya Khovanova |
| R08 T 2  | 38-166  | Zach Abel     |
| R09 T 3  | 38-166  | Zach Abel     |

2:

3:

4:
1. (20 points in total. Each part is worth 5 points.)

Are the following statements below TRUE or FALSE? Give a brief reason.

(a) If $A$ is invertible and $\lambda$ is an eigenvalue of $A$, then $\frac{1}{\lambda}$ is an eigenvalue of $A^{-1}$.

\[ \text{TRUE} \quad \text{if} \quad Ax = \lambda x, \quad \text{then} \quad x = \lambda A^{-1}x \quad \text{and} \quad A^{-1}x = \frac{1}{\lambda} x. \]

(b) If $A$ is a positive definite matrix, then $A^T + I$ is also a positive definite matrix.

\[ \text{TRUE} \quad \text{if} \quad A \text{ is positive def}, \quad \text{then} \quad A^T \quad \text{has eigenvalues of } A \text{ shifted by } +1. \]

(c) The following matrix is positive definite:

\[ A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \]

\[ \text{FALSE.} \quad \text{fails det test} \]

\[ 1 \times 1 \det = 2 \]
\[ 2 \times 2 \det = 3 \]
\[ 3 \times 3 \det = -2 \]

(d) Let $A$ be a real skew-symmetric $n \times n$ matrix, i.e., $A^T = -A$. If $\lambda$ is an eigenvalue of $A$, then so is $-\lambda$.

\[ \text{TRUE} \]

\[ 0 = \det (A - \lambda I) = \det ((A - \lambda I)^T) = \det (A^T - \lambda I) = \det (-A - \lambda I) \]
\[ = (-1)^n \det (A + \lambda I) \]
2. (20 points in total. Each part is worth 10 points.)

(a) Calculate $e^{At}$, where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

$$e^{At} = I + At + \frac{(At)^2}{2!} + \cdots + \frac{(At)^n}{n!} + \cdots$$

$$A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$

(b) Using your answer from part (a), solve the system of differential equations:

$$\frac{du}{dt} = u(t) + v(t),$$

$$\frac{dv}{dt} = 0u(t) + v(t),$$

where $u(1) = 1$ and $v(1) = 0$.

**Warning:** You have been given conditions $u(1) = 1$ and $v(1) = 0$ that are **not** at $t = 0$. Adjust your solution accordingly.

$$\begin{bmatrix} u \\ v \end{bmatrix} = e^{A(t-1)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$e^{A(t-1)} = \begin{bmatrix} e^{t-1} & (t-1)e^{t-1} \\ 0 & e^{t-1} \end{bmatrix}$$

$$\therefore \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} e^{t-1} \\ 0 \end{bmatrix}$$

*Happy for them to spot this solution with part a)*
3. (30 points total. Each part is worth 10 points)

(a) Consider the transformation

\[ T : \{ \text{polynomials of degree } \leq n \} \rightarrow \mathbb{R} \]

given by

\[ T(p) = \int_{0}^{1} p(s)ds. \]

Show that \( T \) is a linear transformation.

\[ T(cp) = \int_{0}^{1} cp(s)ds = c \int_{0}^{1} p(s)ds = cT(p) \]
\[ T(p+q) = \int_{0}^{1} (p+q)(s)ds = \int_{0}^{1} p(s)ds + \int_{0}^{1} q(s)ds \]
\[ = T(p) + T(q) \]

(b) For the linear transformation \( T \) from part (a), you are given the relation

\[ T(x^k) = \int_{0}^{1} x^kdx = \frac{1}{k+1}, \quad k \geq 0. \]

Pick a basis for the input space, a basis for the output space, and find the corresponding matrix that represents \( T \).

**Input space basis:** \( \{1, x, \ldots, x^n\} \)

**Output space basis:** \( \{1\} \)

\[ T(x^k) = \frac{1}{k+1} \]

So,

\[ A = \begin{bmatrix} 1 & \frac{4}{3} & \frac{1}{3} & \frac{1}{4} & \ldots & \frac{1}{n+1} \end{bmatrix} \]
(c) Let $A$ be a $2 \times 2$ matrix such that

$$A \begin{bmatrix} a \\ b \end{bmatrix} = 2 \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix}, \quad A \begin{bmatrix} c \\ d \end{bmatrix} = 2 \begin{bmatrix} a \\ b \end{bmatrix} + 2 \begin{bmatrix} c \\ d \end{bmatrix},$$

where $a, b, c,$ and $d$ are real numbers.

Are $A$ and $\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$ similar? Give a condition on $a, b, c,$ and $d$, if necessary.

$$A \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} c & a \\ d & b \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$$

$$\therefore \quad A = \begin{bmatrix} c & a \\ d & b \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} c & a \\ d & b \end{bmatrix}^{-1}$$

assuming $ad - bc \neq 0$
4. (30 points total. Each part is worth 10 points)

(a) Here is a flowchart for the status of the MIT exam printing machine:

The fractions on the arrows indicate the probability that the machine moves from one particular state to another.

(i) Define the term Markov matrix.
(ii) Why does a Markov matrix always have 1 as an eigenvalue?
(iii) Write down the Markov matrix associated to the flowchart above.

\[ A = \begin{bmatrix} 1/3 & 1/3 & 1 \\ 2/3 & 1/3 & 0 \\ 0 & 1/3 & 0 \end{bmatrix} \]
(b) The printer is upgraded. The associated Markov matrix is now:

\[
A = \begin{bmatrix}
  1/2 & 3/4 & 1 \\
  0 & 1/4 & 0 \\
  1/2 & 0 & 0 \\
\end{bmatrix}.
\]

In the long-run what proportion of time is the printer in each state assuming the printer starts off working?

We just need the eigenvector corresponding to eigenvalue 1:

\[
A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}.
\]

\[
-\frac{1}{2}x + \frac{3}{4}y + z = 0,
\]

\[
-\frac{3}{4}y = 0,
\]

\[
\frac{1}{2}x - z = 0.
\]

\[\therefore y = 0, \quad z = 1, \quad x = 2.\]

So eigenvector is \[
\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.
\]

The printer spends twice as long in State 1 as State 3 in the long-run. It spends no time in State 2.
For a standard color inkjet printer, the associated Markov matrix $A$ has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and linear independent eigenvectors $v_1, v_2, v_3$, where $\lambda_1 = 1$, $\lambda_2 = -1$, and $|\lambda_3| < 1$. What is the long-run behavior of the printer?

$$x_0 = c_1 v_1 + c_2 v_2 + c_3 v_3$$

then

$$A^k x_0 = c_1 \lambda_1^k v_1 + c_2 \lambda_2^k v_2 + c_3 \lambda_3^k v_3$$

For large $k$: $A^k x_0 \approx c_1 v_1 + c_2 (-1)^k v_2$.

"oscillatory state"