Operations Required in Matrix Elimination

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1 Introduction

The amount of time needed to solve $\mathbf{Ax} = \mathbf{b}$ can be estimated by the number of elementary operations performed: addition/subtractions, multiplications and divisions. For some matrices, $\mathbf{Ax} = \mathbf{b}$ is easy to solve. If $\mathbf{A}$ is triangular we can find $\mathbf{x}$ by substitution, and if $\mathbf{A}$ is the identity, then we know $\mathbf{x} = \mathbf{b}$ without performing any operations! However, for practical applications, we are not likely to be so lucky. Given an arbitrary nxn matrix $\mathbf{A}$, how many operations are required to solve $\mathbf{Ax} = \mathbf{b}$?

2 At a Glance

Before we do any calculations, it is helpful to understand where these operations arise. When are additions, subtractions, multiplications and divisions actually performed?

$\mathbf{Ax} = \mathbf{b}$ can be solved in two steps, elimination and back substitution. During elimination, we perform row operations where we replace a row vector r_i with $r_i + cr_j$. This requires one division to calculate c and one multiplication and one addition for each nonzero element of cr_j. During back substitution, we solve equations of the form $a_i x_i + a_{i+1} x_{i+1} + ... + a_{n-1} x_{n-1} + a_n x_n = b_i$ for x_i. This requires one multiplication for each term $a_k x_k$, $k \neq i$, one subtraction to move it to the right hand side, and finally one division to isolate x_i.

One surprising result is that the total number of addition/subtractions must be the same as the total number of multiplications. During elimination and back substitution, whenever a multiplication occurs, it is immediately followed by an addition or a subtraction. Thus, their final counts must be the same.

3 Explicit Calculation

Below we tackle an nxn matrix, let us count the number of operations for a 4x4 matrix:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

3.1 Elimination

The first step is to turn $\mathbf{A}$ into an upper triangular matrix. We perform row operations on the augmented matrix:

$$\mathbf{M} = \left(\begin{array}{cccc|c} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & b_1 \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & b_2 \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & b_3 \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & b_4 \end{array}\right)$$

To eliminate $\mathbf{A}_{2,1}$, we replace row r_2 with $r_2 - a_{2,1}/a_{1,1} * r_1$. Looking at each element of this row vector, we replace $\mathbf{M}_{2,i}$ with $\mathbf{M}_{2,i} - a_{2,1}/a_{1,1} * \mathbf{M}_{1,i}$ for $1 \leq i \leq 5$. This requires 1 division to calculate $(a_{2,1}/a_{1,1})$, 1 multiplication per element to calculate $(a_{2,1}/a_{1,1}) * \mathbf{M}_{1,i}$, and 1 subtraction per element to calculate $\mathbf{M}_{2,i} - (a_{2,1}/a_{1,1} * \mathbf{M}_{1,i})$. In total, this takes 1 division, 5 multiplications and 5 additions. However, we can improve this. By the way we choose to eliminate, we know that $M_{2,1}$ will be 0 without having to calculate $\mathbf{M}_{2,1} - (a_{2,1}/a_{1,1} * \mathbf{M}_{1,1})$. Thus, this takes 1 division, 4 multiplications and 4 additions

Eliminating $\mathbf{A}_{3,1}$ and $\mathbf{A}_{4,1}$ also requires the same number of operations. These three eliminations then take 3 divisions, 12 multiplications and 12 additions and create the matrix:

$$\mathbf{M}' = \left(\begin{array}{cccc|c} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & b_1 \\ 0 & a'_{2,2} & a'_{2,3} & a'_{2,4} & b'_2 \\ 0 & a'_{3,2} & a'_{3,3} & a'_{3,4} & b'_3 \\ 0 & a'_{4,2} & a'_{4,3} & a'_{4,4} & b'_4 \end{array}\right)$$

Continuing elimination on this matrix is the same as performing elimination on the 3x3 matrix:

$$\left(\begin{array}{ccc|c} a'_{2,2} & a'_{2,3} & a'_{2,4} & b'_2 \\ a'_{3,2} & a'_{3,3} & a'_{3,4} & b'_3 \\ a'_{4,2} & a'_{4,3} & a'_{4,4} & b'_4 \end{array}\right)$$

By the same reasoning as above, this will take 2 divisions, 6 multiplications and 6 additions and will leave the 2x2 matrix:

$$\left(\begin{array}{cc|c} a''_{3,3} & a''_{3,4} & b''_3 \\ a''_{4,3} & a''_{4,4} & b''_4 \end{array}\right)$$

Finally, eliminating this matrix takes 1 division, 2 multiplications and 2 additions. All together, it takes 6 divisions, 20 multiplications and 20 additions to get the upper triangular augmented matrix:

$$\mathbf{U} = \left(\begin{array}{cccc|c} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & b_1 \\ 0 & a'_{2,2} & a'_{2,3} & a'_{2,4} & b'_2 \\ 0 & 0 & a''_{3,3} & a''_{3,4} & b''_3 \\ 0 & 0 & 0 & a'''_{4,4} & b'''_4 \end{array}\right)$$

3.2 Back Substitution

To finish solving $\mathbf{Ax} = \mathbf{b}$, we need to calculate $\mathbf{x}$. Starting from the bottom row, we have:

$$a'''_{4,4}x_4 = b'''_4$$

This can be solved with 1 division. The next row is:

$$a''_{3,3}x_3 + a''_{3,4}x_4 = b''_3$$

This requires 1 multiplication to calculate $a''_{3,4}x_4$, 1 subtraction to move that term to right, and 1 division to isolate x_3. Similarly, the second equation:

$$a'_{2,2}x_2 + a'_{2,3}x_3 + a'_{2,4}x_4 = b'_2$$

requires 2 multiplications, 2 subtractions and 1 division, and the first equation:

$$a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + a_{1,4}x_4 = b_1$$

requires 3 multiplications, 3 subtractions and 1 division for a total of 4 divisions, 6 multiplications and 6 addition/subtractions.

Bringing it all together, solving $\mathbf{Ax} = \mathbf{b}$ for a 4x4 matrix takes a total of 10 divisions, 26 multiplications and 26 additions.

3.3 General Case

We can apply the same reasoning for an nxn matrix $\mathbf{A}$. During elimination, the first row will require $[n-1, n(n-1), n(n-1)]$ divisions, multiplications and addition/subtractions respectively. The second row will require $[n-2, (n-1)(n-2), (n-1)(n-2)]$, the third row $[n-2, (n-2)(n-3), (n-2)(n-3)]$, and in general, the i^{th} row requires $[n-i, (n+1-i)(n-i), (n+1-i)(n-i)]$ operations.

During back substitution, the last row requires [1,0,0] operations, the second to last row requires $[1, 1, 1]$ operations, the third to last requires $[1, 2, 2]$ operations, etc. The i^{th} row from the bottom requires $[1, i-1, i-1]$ operations.

There are n rows to the matrix, so i can range from 1 to n. The total number of operations is the sum:

$$\begin{aligned}
&\sum_{i=1}^{n}[n-i,(n+1-i)(n-i),(n+1-i)(n-i)]+\sum_{i=1}^{n}[1,i-1,i-1]\\
=&\sum_{i=1}^{n}[n-i+1,(n+1-i)(n-i)+(i-1),(n+1-i)(n-i)+(i-1)]\\
=&\sum_{i=1}^{n}[-i+(n+1),i^2-(2n)i+(n^2+n-1),i^2-(2n)i+(n^2+n-1)]\\
=&[-n(n+1)/2+n(n+1),n(n+1)(2n+1)/6-(2n)(n(n+1)/2)+n(n^2+n-1),\\
&\ n(n+1)(2n+1)/6-(2n)(n(n+1)/2)+n(n^2+n-1)]\\
=&[n(n+1)/2,(2n^3+3n^2-5n)/6,(2n^3+3n^2-5n)/6]
\end{aligned}$$

In conclusion, solving $\mathbf{Ax}=\mathbf{b}$ takes at most $n(n+1)/2$ divisions, $(2n^3+3n^2-5n)/6$ multiplications, and $(2n^3+3n^2-5n)/6$ addition/subtractions.