SVD practice

(1)

Step1 :

$$A^TA = \begin{bmatrix}1&0\\0&1\\1&0\\0&1\end{bmatrix}\begin{bmatrix}1&0&1&0\\0&1&0&1\end{bmatrix} = \begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{bmatrix}$$

eigenvalues are 0, 2,

$$A^TA\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \quad A^TA\begin{bmatrix}0\\1\\0\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \quad A^TA\begin{bmatrix}0\\1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\2\\0\\2\end{bmatrix}, \quad A^TA\begin{bmatrix}1\\0\\1\\0\end{bmatrix} = \begin{bmatrix}2\\0\\2\\0\end{bmatrix}$$

$$\therefore V = \begin{bmatrix}1/\sqrt2&0&-1/\sqrt2&0\\0&1/\sqrt2&0&1/\sqrt2\\1/\sqrt2&0&1/\sqrt2&0\\0&1/\sqrt2&0&-1/\sqrt2\end{bmatrix} \qquad \Sigma = \begin{bmatrix}2&0&6&0\\0&2&0&0\end{bmatrix}$$

Step 2 :

$$AA^T = \begin{bmatrix}2&0\\0&2\end{bmatrix}, \quad \text{eigenvalues } 2, \quad AA^T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}, \quad AA^T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix}$$

$$U = \begin{bmatrix}\cancel{2}1&0\\0&1\end{bmatrix}$$

step 3 :

$$A = \begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}2&0&0&0\\0&2&0&0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt2}&0&-1/\sqrt2&0\\0&1/\sqrt2&0&1/\sqrt2\\1/\sqrt2&0&1/\sqrt2&0\\0&1/\sqrt2&0&-1/\sqrt2\end{bmatrix}^T$$

Signs work :).

2. This is very messy. Let me just write down the result:

$$A = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ \sqrt{2/3} & -1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} +1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \\ \sqrt{3}/2 & 0 & -1/2 \\ 1/2\sqrt{3} & \sqrt{2/3} & 1/2 \end{bmatrix}$$

In the final it will <u>not</u> be as messy as this.

3. one ~~sig~~ singular value.

$$A = \sigma_1 u_1 v_1^T = [1]\,[0\ 1\ 1\ 0\ \pi\ 6]$$

$$= \sqrt{\ell}\ [1]\,[0\ 1\ 1\ 0\ \pi\ 6]/\sqrt{\ell}$$

$$\ell = 1^2+1^2+\pi^2+6^2 = 38+\pi^2.$$

$$\therefore\ \sigma_1 = \sqrt{\pi^2+38}$$

4.

$$\sigma_n \|\underline{x}\| \le \|A\underline{x}\| \le \sigma_1 \|\underline{x}\|$$

LOWER BOUND ATTAINED WITH $\underline{x} = \underline{v}_n$ ← last col. of V

UPPER BOUND ATTAINED WITH $\underline{x} = \underline{v}_1$ ← 1^{st} col. of V.

5.

$$A = \begin{bmatrix} -1/3 & 2/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T$$

$$C(A) = \left\{ \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix} \right\} \qquad N(A) = \left\{ \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \right\}$$

$$C(A^T) = \left\{ \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \right\} \qquad N(A^T) = \left\{ \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix} \right\}$$

6 i)

$$Q = Q I_n I_n^T$$

(ii)

$$D = I_n D I_n^T$$

(iii)

$$A = U \Sigma_1 V^T$$

not sure if $V = U$ or not.

But we do knows that $V = UD$, where $D = \text{diag}(\pm 1, \pm 1, \dots \pm 1)$.

7. If A is +ve definite, then

$$A = Q \Lambda Q^T$$ is the SVD (~~a~~ assume Λ has diagonal entries ordered so that $\lambda_1 \geq \dots \geq \lambda_n \geq 0$).

If A is −ve definite then

$$A = Q \Lambda Q^T$$ is not the SVD

But we do know: $A = U \Sigma V^T$, where $\Sigma = -\Lambda$ (assuming diag entries are ordered)

$$U = Q$$
$$V = -Q$$

is the SVD.