Pset 3 solutions

1.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{matrix} (2) \leftarrow (2) - (1) \\ (3) \leftarrow (3) - 4(1) \end{matrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -2 & -3 \end{bmatrix} (3) \leftarrow (3) + (2)$$

$$\longrightarrow \begin{bmatrix} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

free-columns correspond to free-variables

Now reduce further:

$$\longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} (1) \leftarrow (1) - (2)$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row-reduced echelon form.

(b) A solution to $\underline{A}\underline{x} = \underline{b}$ is $\underline{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore$ All solutions are $= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + y, \quad y \in N(A)$

Here,

$$N(A) = c \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

c)

	$N(A) = \{[0]\}$	$N(A) \neq \{[0]\}$
$b \in C(A)$	Unique solution	Infinitely many solutions.
$b \notin C(A)$	No solution	No solution

2 a) Basis : 3x3

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

In general: for $n \times n$: $\cancel{B_{ii}}$ if $\cancel{B}$ B_{ij} = zero matrix with (i,j) entry $= 1$.

Then Basis $= \{ B_{ij}, 1 \le i \le j \le n \}$

(b) Symmetric 3x3

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

~~(c) No not a vector space. Elimination matrices have ones on diagonal.~~

(c) No, permutation matrices do not form a vector space.

3 (a) FALSE ~~No~~, row manipulations do not preserve $C(A^T)$ only $C(A)$.

(b) TRUE, $\dim(C(A^T)) = \dim(C(A))$.

(c) ~~FALSE~~ TRUE, any example is fine.

4.

$$\begin{bmatrix} -2 & 4 & 6 \\ 2 & -4 & 3 \\ 1 & -2 & 1 \end{bmatrix} \begin{matrix} \textcircled{2} \leftarrow \textcircled{2} + \textcircled{1} \\ \textcircled{3} \leftarrow \textcircled{3} + \frac{1}{2}\textcircled{1} \end{matrix} \longrightarrow \begin{bmatrix} -2 & 4 & 6 \\ 0 & 0 & 9 \\ 0 & 0 & 4 \end{bmatrix} \textcircled{3} \leftarrow \textcircled{3} - \frac{4}{9}\textcircled{2}$$

(not done... two rows down here)

$$\longrightarrow \begin{bmatrix} \boxed{-2} & 4 & 6 \\ 0 & 0 & \boxed{9} \\ 0 & 0 & 0 \end{bmatrix}$$

$C(A) = c\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} + d\begin{bmatrix} 6 \\ 3 \\ +1 \end{bmatrix}$ and basis is $\left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix} \right\}$.

$N(A) = c\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and basis is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

(b) $\dim(C(A))$ = # pivot cols = 2 $\quad \therefore \quad \dim(C(A)) + \dim(N(A)) = 3$

$\dim(N(A))$ = # free-variables = 1

= # free-columns.

(c) When doing "elimination++" to get A into echelon form, ~~the~~

a column of A is either: (1) A successful pivoting column

OR (2) A complete failure, or free-column.

Since $\dim(C(A))$ = # pivot cols

$\dim(N(A))$ = # free-cols

we have $\dim(C(A)) + \dim(N(A))$ = # columns of A.