Your name: 

Recitation: 

<table>
<thead>
<tr>
<th>problem</th>
<th>score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>/15</td>
</tr>
<tr>
<td>2</td>
<td>/15</td>
</tr>
<tr>
<td>3</td>
<td>/15</td>
</tr>
<tr>
<td>4</td>
<td>/15</td>
</tr>
<tr>
<td>5</td>
<td>/15</td>
</tr>
<tr>
<td>6</td>
<td>/15</td>
</tr>
<tr>
<td>7</td>
<td>/10</td>
</tr>
<tr>
<td>total</td>
<td>/100</td>
</tr>
</tbody>
</table>
Problem 1 (15 points):

A matrix $A = LU$ has the LU factors

$$L = \begin{pmatrix} 1 & -1 & -2 & 0 \\ -2 & 1 & 0 & -2 \\ 0 & -2 & 1 & 1 \\ -1 & -1 & -2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 & -2 & 0 \\ 1 & 0 & -2 \\ 1 & 0 & 1 \end{pmatrix}$$

(a) If $b = \begin{pmatrix} -1 \\ 2 \\ 2 \\ -4 \end{pmatrix}$, what is $x = A^{-1}b$?

(b) Assuming you solved the previous part efficiently, roughly how much more arithmetic operations would be required for the same approach if the matrices were $8 \times 8$ instead of $4 \times 4$? It should be about ______ times more.

(c) If you form a new $4 \times 5$ matrix $B = (A \ b)$ by appending the vector $b$ (from above) as an extra column after $A$, and perform the same elimination steps as were used to get the LU factors above, what upper-triangular matrix would you obtain? (Hint: if you did part (a) properly, this part can be done with \textit{no arithmetic}.)

2
(blank page for your work if you need it)
Problem 2 (15 points):

You are given the recurrence relation

\[(2I + B^T B)x_{n+1} = (2I - B^T B)x_n\]

where \(B\) is a real \(5 \times 3\) matrix. We start with a vector \(x_0\) and compute \(x_1, x_2, \ldots\).

(a) \(x_n = A^n x_0\) for some matrix \(A\) (independent of \(x_0\)). What is \(A\)?

(b) If \(\lambda\) is an eigenvalue of \(B^T B\), give an eigenvalue of \(A\).

(c) Circle all possible behaviors of \(x_n\) for large \(n\), given the information above: decaying to zero, oscillating but not growing or decaying in length, going to a nonzero constant vector, or growing longer and longer. Explain your answers by giving some property (or properties) that must be true of the eigenvalues of \(A\).

(d) If \(x_1 = 0\) for a nonzero \(x_0\), give one of the singular values (\(\sigma\)) of \(B\).
Problem 3 (15 points):

The distance between a point $b$ and a plane in $\mathbb{R}^3$ is defined as the minimum distance $\|b - y\|$ between $b$ and any point $y$ in the plane.

(a) Suppose the points $y$ in the plane are of the form $y = c + \alpha a_1 + \beta a_2$ for all real numbers $\alpha$ and $\beta$, given vectors $c, a_1, a_2 \in \mathbb{R}^3$ that define the plane ($a_1$ and $a_2$ are linearly independent). Under what condition(s) on $c, a_1, a_2$ is the plane a subspace of $\mathbb{R}^3$?

(b) Write down a $2 \times 2$ system of equations, in terms of the vectors $a_1, a_2, c, b$ (or matrices defined from these vectors) whose solution gives the $(\alpha \beta)$ for the closest point $y$ in the plane to $b$.

(c) For this closest point $y$, $b - y$ is in what subspace of the matrix $A = (a_1a_2)$? What is the dimension of this subspace?

(d) For $a_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $a_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, find a vector $d$ such that the distance between any point $b$ and the plane is equal to $|d^T(b-c)|$. What subspace of $A$ contains $d$?
Problem 4 (15 points):

You are given the following matrix:

\[ A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & -3 & 4 \\ 1 & -2 & 1 & -4 \end{pmatrix} \]

(a) Find the complete solution \( x \) (i.e. all solutions) to \( Ax = b \) for \( b = \begin{pmatrix} 3 \\ 9 \\ -4 \end{pmatrix} \).

(b) \( A^T y = d \) is solvable if and only if \( \begin{pmatrix} 3 \\ 9 \\ -4 \end{pmatrix}^T z = 0 \) for some \( z \). Give such a vector \( z \).
(blank page for your work if you need it)
Problem 5 (15 points):

QR factorization of the matrix $A$ (e.g. via Gram–Schmidt) yields $A = QR$, where

$$Q = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}, \quad R = \begin{pmatrix}
1 & 2 & 0 \\
1 & 2 \\
0 & 2
\end{pmatrix}.$$

(a) Which columns of $A$ were orthogonal to begin with, if any?

(b) What is the orthogonal projection $p$ of the vector $b = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ onto $C(A)$?

(c) If we are minimizing $\|Ax - b\|$ (i.e. solving the least-square problem) for $b = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, you should be able to quickly get an upper-triangular system of equations $U\hat{x} = c$ for the least-square solution $\hat{x}$. What are the upper-triangular matrix $U$ and the right-hand-side vector $c$?
Problem 6 (15 points):

You are given the nonsymmetric, diagonalizable matrix

\[
A = \begin{pmatrix}
-1 & 1 & 3 \\
1 & -3 & -2 \\
-1 & 0 & -3
\end{pmatrix}.
\]

and we want to understand the solutions of the ODE

\[
\frac{dx}{dt} = Ax
\]

for some initial condition \(x(0)\).

(a) Show (by any test you want, e.g. the pivot test) that the matrix

\[
A + A^H = \begin{pmatrix}
-2 & 2 & 2 \\
2 & -6 & -2 \\
2 & -2 & -6
\end{pmatrix}
\]

is negative definite.

(b) If \(Av = \lambda v\) is an eigensolution of \(A\) (\(v\) and \(\lambda\) may be complex), look at \(v^H (A + A^H) v\) and use the fact that \(A + A^H\) is negative definite to show that the real part of \(\lambda\) must be negative.

(c) What can you conclude from the previous parts about the solutions \(x(t)\) as \(t \to \infty\)?

(d) If \(A + A^H\) is negative definite (so that \(A\)'s eigenvalues have negative real parts), but \(A\) is defective, does your answer to the previous part about \(x(\infty)\) change? Why or why not?
Problem 7 (10 points):
The following parts can be answered independently (and refer to different matrices). Little or no calculation should be needed.

(a) If $C(B)$ is a subspace of $N(A)$, then either (circle one) $AB$ or $BA$ must be simply ______.

(b) If $A$ is a real-symmetric $3 \times 3$ matrix with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$ and corresponding real eigenvectors $v_1, v_2, v_3$, then an explicit equation for $A^{-1}b$ in terms of sums/products involving these eigenvectors and $b$, with no matrix inverses, is:

(c) If $A$ is a $3 \times 3$ non-singular real matrix with singular values $\sigma_1, \sigma_2, \sigma_3$, then give formulas in terms of $\sigma_1, \sigma_2, \sigma_3$ for $\det(A^T A) =$ ________ and $\det(A) =$ ________.

(d) If $N(A)$ is spanned by the vector $v \neq 0$, then projection matrices onto two of the fundamental subspaces of $A$ are:

\[ \text{and} \]

(write down two matrices and indicate which subspaces they project onto).

(e) If $A$ is similar to the matrix \[
\begin{pmatrix}
3 & 6 & 2 \\
17 & 3 & \\
4 & & \\
\end{pmatrix}
\], then the eigenvalues of $A$ are: ________________.
(blank page for your work if you need it)