1. (12 pts.) Let

\[
A = \begin{bmatrix}
7 & 0 & 2 & 4 \\
7 & 1 & 3 & 6 \\
14 & -1 & 3 & 6 \\
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
2 & -1 & 1 \\
\end{bmatrix} \begin{bmatrix}
7 & 0 & 2 & 4 \\
0 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

(a) Find bases for the four fundamental subspaces.

(b) Find the conditions on \(b_1, b_2, \) and \(b_3\) so that

\[
Ax = \begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
\end{bmatrix}
\]

has a solution.

(c) If \(Ax = b\) has a solution \(x_p\), describe all of the solutions.
2 (10 pts.) Let $A$ and $B$ be any two matrices so that the product $AB$ is defined.

(a) Explain why every column of $AB$ is in the column space of $A$.

(b) How does part (a) lead to the conclusion that the rank of $AB$ is less than or equal to the rank of $A$? State your reasoning in logical steps.
3 (10 pts.) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation satisfying

$$T \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -4 \\ 3 \end{bmatrix} \quad \text{and} \quad T \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -10 \\ 8 \end{bmatrix}.$$ 

(a) Find $T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$.

(b) What is the matrix $A$ expressing $T$ in terms of the standard basis vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$? (The same basis is used for the input and the output.)

(c) What is the matrix $B$ expressing $T$ in terms of the basis consisting of eigenvectors of $A$? (The same basis is used for the input and output.) (There are two possible correct answers, depending on what order you pick the eigenvectors.)
4 (16 pts.) Let \( V \) be the subspace of \( \mathbb{R}^3 \) consisting of vectors 
\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]
satisfying
\[x + 2y - 5z = 0.\]

(a) Find a \( 3 \times 2 \) matrix \( A \) whose column space is \( V \).

(b) Find an orthonormal basis for \( V \).

(c) Find the projection matrix \( P \) projecting onto the left nullspace (not the column space!) of \( A \).

(d) Find the least squares solution to
\[
Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.
\]
5 \textbf{(15 pts.)} Suppose

\[
Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{has no solution}
\]

but

\[
Ax = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \text{has infinitely many solutions.}
\]

(a) Find all possible information about \( r, m, \) and \( n \). (The rank and the shape of \( A \).)

(b) Find an example of such a matrix \( A \) with \( r, m, \) and \( n \) all as small as possible.

(c) How do you know that \[
\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}
\]
is not in the nullspace of \( A^T \)?
6 (13 pts.) In each case give all the information you can about the eigenvalues and eigenvectors, when the matrix \( A \) has the following property:

(a) The powers \( A^k \) approach the zero matrix.

(b) The matrix is symmetric positive definite.

(c) The matrix is not diagonalizable.

(d) The matrix has the form \( A = uv^T \), where \( u \) and \( v \) are vectors in \( \mathbb{R}^3 \).
   (You might want to try an example.)

(e) \( A \) is similar to a diagonal matrix with diagonal entries 1, 1, and 2.
7 (12 pts.) Define a sequence of numbers in the following way: \( G_0 = 0, \ G_1 = 1/2, \) and 
\[ G_{k+2} = \frac{(G_{k+1} + G_k)}{2}. \] (Each number is the average of the two previous numbers.)

(a) Set up a \( 2 \times 2 \) matrix \( A \) to get from 
\[
\begin{bmatrix}
G_{k+1} \\
G_k
\end{bmatrix}
\] to 
\[
\begin{bmatrix}
G_{k+2} \\
G_{k+1}
\end{bmatrix}
\].

(b) Find an explicit formula for \( G_k \).

(c) What is the limit of \( G_k \) as \( k \to \infty \)?
8 (12 pts.) (a) Suppose $A$ is a $4 \times 4$ matrix of rank 3, and let

$$x = \begin{bmatrix}
C_{11} \\
C_{12} \\
C_{13} \\
C_{14}
\end{bmatrix}$$

be the cofactors of its first row. Explain why $Ax = 0$. (So the cofactors give a formula for a nullspace vector!)

Hint: The first component of $Ax$ and the second component of $Ax$ are determinants of (different) matrices. What are these matrices and why do they have zero determinants? (The 3rd and 4th components of $Ax$ follow similarly, so you can just answer for the 1st and 2nd components.)

(b) Compute the determinant of

$$B = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1
\end{bmatrix}.$$  

Hint: You might find it convenient to use the fact that the columns are orthogonal.