1. (25 pts.) Suppose that row operations (elimination) reduce the matrices $A$ and $B$ to the same row echelon form

$$R = \begin{bmatrix} 1 & 2 & 0 & 7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}. $$

(a) Which of the four subspaces are sure to be the same for $A$ and $B$? ( 
$C(A) = C(B)$? $N(A) = N(B)$? $C(A^T) = C(B^T)$? $N(A^T) = N(B^T)$?)

(b) Each time the subspaces in part (a) are the same for $A$ and $B$, find a basis for that subspace.

(c) True or False (A is any matrix and $x$, $y$ are two vectors): If $Ax$ and $Ay$ are linearly independent then $x$ and $y$ are linearly independent.
2 (25 pts.) Suppose

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
7 & -1 & 2
\end{bmatrix} \begin{bmatrix}
1 & 0 & 1 & 4 & 5 \\
0 & 1 & 2 & 2 & 1 \\
0 & 0 & 0 & 1 & 1
\end{bmatrix}
\]

(a) Find a basis for the nullspace of \( A \).

(b) Find a basis for the column space of \( A \).

(c) Give the complete solution to

\[
Ax = \begin{bmatrix}
3 \\
3 \\
21
\end{bmatrix}
\]
Suppose $A$ is a $3 \times 5$ matrix and the solutions to $A^T y = 0$ are spanned by the vectors

\[
y = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.
\]

(a) What is the rank of this $A$?

(b) For all $A$, why does the rank of $A$ equal the rank of the block matrix

\[
B = \begin{bmatrix} A & A \\ A & A \end{bmatrix}?
\]

(c) If the rank of a matrix $A$ equals the number of rows ($r = m$), what do we know about the equation $Ax = b$?
4 (25 pts.) Suppose $A$ is a 4 by 3 matrix, and the complete solution to

$$Ax = \begin{bmatrix} 1 \\ 4 \\ 1 \\ 1 \end{bmatrix} \quad \text{is} \quad x = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$ 

(a) What is the third column of $A$?

(b) What is the second column of $A$?

(c) Give all known information about the first column of $A$. 