1 (36 pts.) The differential equation is
\[ \frac{du}{dt} = Au \quad \text{with} \quad A = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix} \quad \text{and} \quad u(0) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}. \]

(a) Find the eigenvalues and eigenvectors and diagonalize to \( A = SAS^{-1} \).

\( A \) is not invertible, hence one eigenvalue is 0.
\( Tr(A) = -5 \), so the other eigenvalue of \( A \) must be \(-5\).
An eigenvector of \( A \) with eigenvalue 0 is \((3, 2)\).
An eigenvector of \( A \) with eigenvalue \(-5\) is \((1, -1)\).

\[ A = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 1/5 & 1/5 \\ 2/5 & -3/5 \end{bmatrix}. \]

(b) Solve for \( u(t) \) starting from the given \( u(0) \).

General solution is \( u(t) = c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \)

The condition \( u(0) = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \) is satisfied when \( c_1 = 1, c_2 = 2 \).
(c) Compute the matrix $e^{At}$ using $S$ and $\Lambda$.

$$A = SAS^{-1} \Rightarrow e^{At} = Se^{\Lambda t}S^{-1}.$$ So

$$e^{At} = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-5} \end{bmatrix} \begin{bmatrix} 1/5 & 1/5 \\ 2/5 & -3/5 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2e^{-5t} + 3 & -3e^{-5t} + 3 \\ -2e^{-5t} + 2 & 3e^{-5t} + 2 \end{bmatrix}$$

(d) As $t$ approaches infinity, find the limits of $u(t)$ and $e^{At}$.

As $t \rightarrow \infty$, $e^{-5t} \rightarrow 0$, $u(t) \rightarrow \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, and $e^{At} \rightarrow \frac{1}{5} \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}$
2 (40 pts.) The matrix $A$ has 3’s on the diagonal and 2’s everywhere else:

$$A = \begin{bmatrix}
3 & 2 & 2 \\
2 & 3 & 2 \\
2 & 2 & 3 \\
\end{bmatrix}$$

(a) Decide if $A$ is positive definite. What is the minimum value of $x^T A x$ for all vectors $x$ in $\mathbb{R}^3$?

$A$ is positive definite since:

1. $A$ is symmetric,
2. The upper left determinants are 3, 5, 7.

$A$ is positive definite $\Rightarrow x^T A x \geq 0$.

When $x = 0$, $x^T A x = 0$. So 0 is the minimum.

(b) All entries of $B = A - I$ are 2’s. From its rank find all the eigenvalues of $B$ and then all the eigenvalues of $A$.

All three columns of $B$ are the same

$\Rightarrow Rank(B) = 1$

$\Rightarrow N(B)$ has dimension 2

$\Rightarrow B$ has two independent eigenvectors with eigenvalue 0

$\Rightarrow \lambda_1 = \lambda_2 = 0$, and $\lambda_3 = tr(B) = 6$.

The eigenvalues of $A$ are 1, 1, 7.

(c) Write down any one specific symmetric matrix $C$ that is similar to $A$.

Write down if possible any one nonsymmetric matrix $N$ that is similar to $A$. Write down a matrix $J$ with the same eigenvalues as $A$ that is not similar to $A$. (Give the 9 numbers in $C, N, J$.)

$$C = \Lambda = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 7 \\
\end{bmatrix}, \quad N = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 7 \\
\end{bmatrix}, \quad J = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 7 \\
\end{bmatrix}.$$
Explanation: $A$ is symmetric (so you could have let $C = A$) and hence can be diagonalized to $\Lambda$. Consider

$$D = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 7 \end{bmatrix}$$

$D$ has eigenvalues 1, 1, 7.

If $D$ is similar to $A$, then $D$ can also be diagonalized to $\Lambda$ and hence must have two eigenvectors with eigenvalue 1. This is possible iff $x = 0$. Our choice of $C$ is obtained by letting $y = z = 0$. Our choice of $N$ is obtained by letting $y = 0$, $z = 1$.

When $x \neq 0$, $D$ is not similar to $A$. Our choice of $J$ is obtained by letting $x = 1$.

Note: There are choices for $C$, $N$, and $J$ which are not upper triangular.

(d) For the 6 by 6 matrix $A_6$ with 3’s on the diagonal and 2’s everywhere else use the same method (with $A_6 - I$) to find the six eigenvalues. If you make a good choice of eigenvectors, in what form can you factor $A$?

The matrix $B_6 = A_6 - I$ has rank 1, so it has 5 independent eigenvectors with eigenvalue 0. It follows that the eigenvalues of $B_6$ are 0, 0, 0, 0, 0, 12 = $tr(B)$ and the eigenvalues of $A_6$ are 1, 1, 1, 1, 1, 13.

We already know that $A = SAS^{-1}$ for some $S$ whose columns are independent eigenvectors of $A$. But $A$ is symmetric, so we can choose its eigenvectors to be orthonormal and have $A = QAQ^{-1}$, where $Q$ is orthogonal.
Suppose \( A = U \Sigma V^T = (\text{orthogonal } 2 \times 2) \) (diagonal) (\text{orthogonal } 3 \times 3) 

\[
U = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad V = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}
\]

(a) What are the eigenvalues and eigenvectors of \( A^T A \)?

\[
A^T A = (U \Sigma V^T)^T (U \Sigma V^T) = V \Sigma^T \Sigma V^T.
\]

\[
\Sigma^T \Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

The eigenvalues of \( A^T A \) are 16, 1, 0.

\( v_1 \) is an eigenvector of \( A^T A \) with eigenvalue 16.

\( v_2 \) is an eigenvector of \( A^T A \) with eigenvalue 1.

\( v_3 \) is an eigenvector of \( A^T A \) with eigenvalue 0.

(b) What is the nullspace of \( A \)? (Describe the whole nullspace.)

The nullspace of \( A \) is the linear span of \( v_3 \).

(c) What is the row space of \( A \)? (Describe the whole row space.)

The row space of \( A \) is the linear span of \( v_1, v_2 \).