18.06 Solutions to Midterm Exam 2, Spring, 2001

1. (40pts.) Consider the matrix

\[ A = \begin{pmatrix}
1 & 0 & -1 \\
3 & 1 & -1 \\
9 & 5 & 1 \\
9 & 8 & 7
\end{pmatrix} \]

(a) Find the rank of \( A \).

- After doing row operations on the matrix \( A \), we obtain

\[ A = \begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & 2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \]

Since there are two non-zero pivots, \( A \) has rank 2.

(b) Find a basis for the row space of \( A \), and find a basis for the nullspace of \( A \). What is the dimension of the nullspace of \( A \)?

- A basis for the row space of \( A \) is given by the two pivot rows: the first two rows, since we did not have to exchange rows during row operations; \{(1, 0, -1), (3, 1, -1)\}.

The solutions of \( Ax = 0 \) are given by \( x = \alpha(1, -2, 1)^T \) where \( \alpha \in \mathbb{R} \). Hence, a basis for the nullspace is \((1, -2, 1)^T\). Since the basis of the nullspace contains one vector, \( \dim N(A) = 1 \).

(c) What can you say about the relation between the rank and the dimension of the nullspace of \( A \)?

- \( \dim N(A) + \text{rk}(A) = \text{number of columns of } A \). Here, \( 1 + 2 = 3 \).

(d) Verify that all vectors in your basis of the nullspace are orthogonal to all vectors in your basis of the row space.

- \((1, 0, -1) \cdot (1, -2, 1) = 0\), and \((3, 1, -1) \cdot (1, -2, 1) = 0\).
2. (30pts.) Let \(a, b \in \mathbb{R}\), and let
\[
A = \begin{pmatrix}
1 & 2 & 3 & a \\
1 & 0 & -1 & 0 \\
0 & 1 & 2 & b
\end{pmatrix}.
\]

(a) What are the dimensions of the four subspaces associated with the matrix \(A\)? This will of course depend on the values of \(a\) and \(b\), and you should distinguish all different cases.

- After doing row operations on the matrix \(A\), we find
\[
A = \begin{pmatrix}
1 & 2 & 3 & a \\
0 & 2 & 4 & a \\
0 & 0 & 0 & a - 2b
\end{pmatrix}.
\]

If \(a = 2b\), then there are two non-zero pivots, and so \(\text{rk}(A) = \dim \text{col}(A) = \dim \text{row}(A) = 2\). Also, \(\dim \text{null}(A) = 4 - 2 = 2\) and \(\dim \text{left-null}(A) = 3 - 2 = 1\).

If \(a \neq 2b\), then there are three non-zero pivots, and so \(\text{rk}(A) = \dim \text{col}(A) = \dim \text{row}(A) = 3\). Also, \(\dim \text{null}(A) = 4 - 3 = 1\) and \(\dim \text{left-null}(A) = 3 - 3 = 0\).

(b) For \(a = b = 1\), give a basis for the column space of \(A\). Is this also a basis for \(\mathbb{R}^3\)? Justify your answer.

- A basis is given by the columns of \(A\) which lead to non-zero pivots,
\[
\text{basis} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}
\]

Since \(\mathbb{R}^3\) has dimension 3, and these are 3 linearly independent vectors in \(\mathbb{R}^3\), they are indeed a basis for \(\mathbb{R}^3\).
3. (30pts.) An experiment at the seven times \( t = -3, -2, -1, 0, 1, 2, 3 \) yields the consistent result \( b = 0 \), except at the last time \( (t = 3) \), when we get \( b = 28 \). We want the best straight line \( b = C + Dt \) to fit these seven data points by least squares.

(a) Write down the equation \( Ax = b \) with unknowns \( C \) and \( D \) that would be solved if a straight line exactly fit the data.

\[
\begin{pmatrix}
1 & -3 \\
1 & -2 \\
1 & -1 \\
1 & 0 \\
1 & 1 \\
1 & 2 \\
1 & 3
\end{pmatrix}
\begin{pmatrix}
C \\
D
\end{pmatrix} = 
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
28
\end{pmatrix}
\]

(b) Use the method of least squares to find the best fit values for \( C \) and \( D \).

\[
A^T A = \begin{pmatrix} 7 & 0 \\ 0 & 28 \end{pmatrix}, \quad A^T b = \begin{pmatrix} \frac{28}{4} \end{pmatrix}.
\]

Solving \((A^T A)x = (A^T b)\), we find that \( x_{\text{sol}} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \).

(c) This problem is really that of projecting the vector \( b = (0,0,0,0,0,28)^T \) onto a certain subspace. Give a basis for that subspace, and give the projection \( p \) of \( b \) onto that subspace.

- A basis is given by the columns of \( A \), \( \{(1,1,1,1,1,1)^T, (-3,-2,-1,0,1,2,3)^T\} \). The projection is given by

\[
p = Ax_{\text{sol}} = \begin{pmatrix} -5 \\ -2 \\ 1 \\ 3 \\ 7 \\ 10 \\ 13 \end{pmatrix}.
\]