18.06 Problem Set 6

Due Wednesday, April 24

Problem 1. A city is served by two newspapers, the Star and the Times. Each year the Star loses 40% of its subscribers to the Times and retains 60% of its subscribers. During the same time period, the Times loses 10% of its subscribers to the Star while retaining the other 90%.

a) Write down a Markov matrix that describes the transition of subscribers between the two papers each year.

b) Find the steady state vector for the matrix in (a).

c) After many years, approximately what percentage of the subscribers will subscribe to the Times?

Problem 2.

a) Let U and V be unitary matrices. Show that U^{-1} and UV are unitary.

b) Why is the determinant of a Hermitian matrix a real number?

Problem 3. Suppose A is a square matrix with eigenvalues 1 and $\frac{1}{3}$ and corresponding eigenvectors $\mathbf{v_1}$ and $\mathbf{v_2}$. Consider the relation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ for integers $k \geq 1$, and $\mathbf{x}_0 = 2\mathbf{v_1} + 5\mathbf{v_2}$.

a) Find a formula for $\mathbf{x}_k$ in terms of the eigenvectors above.

b) To what limit does $\mathbf{x}_k$ tend as k tends to infinity?

Problem 4. Let $r(t)$ and $w(t)$ denote the rabbit and wolf populations in a particular area at time t. They change with respect to time according to the differential equations

$$\frac{dr}{dt} = 10r - 3w \quad , \quad \frac{dw}{dt} = 5r + 2w.$$

Find the functions $r(t)$ and $w(t)$ using the methods of section 6.3, and assuming that

$$r(0) = 30 = w(0)$$

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Problem 5. Suppose A and B are $n \times n$ matrices with the properties that $AB = BA$ and $N(A) = N(B)$ (the nullspaces are the same).

a) Show that if $\mathbf{v}$ is an eigenvector for A corresponding to the non-zero eigenvalue λ, then $B\mathbf{v}$ is also a λ-eigenvector for A.

b) Suppose that A has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, with corresponding eigenvectors $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$. Show that B is diagonalizable. (*Hint: Show that A and B share the same eigenvectors. What is a basis for the λ_i-eigenspace?*)