18.06 Problem Set #6 Solutions

1. a) If we denote the number of subscribers to Star at k-th year by s_k, and the number of subscribers to Times by t_k, then the conditions in the problems tell us $s_{k+1} = .6s_k + .1t_k$, $t_{k+1} = .4s_k + .9t_k$. So the Markov matrix describing transition of subscribers is $M = \begin{bmatrix} .6 & .1 \\ .4 & .9 \end{bmatrix}$.

   b) The characteristic equation of M is $\lambda^2 - 1.5\lambda + .5 = 0$. So the eigenvalues of M are 1 and 0.5. The steady state vector, which is a eigenvector corresponding to eigenvalue 1, is $(1, 4)$.

   c) The formula for (s_k, t_k) is, $(s_k, t_k) = c_1 1^k(1, 4) + c_2(.5)^k(1, -1)$, where c_1, c_2 are some constants. When k goes to infinity, $(.5)^k$ goes to zero. So (s_k, t_k) goes to $c_1(1, 4)$. This means after many years, approximately 80% of the subscribers will subscribe to the Times.

2. a) $\overline{(UV)}^T(UV) = \overline{V}^T\overline{U}^TUV = \overline{V}^TV = I$, so UV is unitary if both U and V are. In the first step, we basically use the associativity of matrix multiplication;in the second step we use the fact that U is unitary;in the third step we use the fact that V is unitary. $\overline{U^{-1}}^T = (\overline{U}^T)^{-1} = (U^{-1})^{-1}$, so U^{-1} is unitary if U is. In the first step we use the fact that we can commute the operation of taking complex conjugation, taking transpose or taking inverse of a matrix; in the second step we use the fact U is unitary, so $\overline{U}^T = U^{-1}$.

   b) Suppose A is a hermitian matrix, i.e. $\overline{A}^T = A$. Then $\det(\overline{A}^T) = \det(\overline{A}) = \overline{\det(A)}$. But $\det(\overline{A}^T) = \det(A)$ because A is hermitian. So $\overline{\det(A)} = \det(A)$, $\det(A)$ has to be a real number.

3. a) $\mathbf{x}_k = A^k\mathbf{x}_0 = A^k(2\mathbf{v}_1 + 5\mathbf{v}_2) = 2A^k\mathbf{v}_1 + 5A^k\mathbf{v}_2 = 2\mathbf{v}_1 + 5(\frac{1}{3})^2\mathbf{v}_2$, The last step is because $\mathbf{v}_1$, $\mathbf{v}_2$ are eigenvectors, use $A\mathbf{v}_i = \lambda_i\mathbf{v}_i$ iteratively, one can have $A^k\mathbf{v}_i = \lambda_i^k\mathbf{v}_i$. Here $\lambda_1 = 1, \lambda_2 = 1/3$.

   b) In above formula for $\mathbf{x}_k$, when k tends to infinity, $(\frac{1}{3})^k$ tends to zero. So $\mathbf{x}_k$ tends to $2\mathbf{v}_1$.

4. The coefficient matrix for the ODE is $A = \begin{bmatrix} 10 & -3 \\ 5 & 2 \end{bmatrix}$. First we find the characteristic equation $\det(\lambda I - A) = 0$, i.e. $\lambda^2 - 12\lambda + 35 = 0$. So the two eigenvalues are 5 and 7.

   Substitute λ by 5 or 7 respectively into the matrix $\lambda I - A$ and solve for $(\lambda I - A)\mathbf{v} = 0$ to find eigenvectors. The eigenvector associated to 5 is $\mathbf{v}_1 = (3, 5)$, and eigenvector associated to 7 is $\mathbf{v}_2 = (1, 1)$.

   The general solution to the differential equation is $(r(t), w(t)) = c_1e^{5t}\mathbf{v}_1 + c_2e^{7t}\mathbf{v}_2$. Let $t = 0$ and use the condition $r(0) = 30 = w(0)$, we find $c_1 = 0, c_2 = 30$. So the final solution is $r(t) = w(t) = 30e^{7t}$

5. a) If $Av = \lambda v$ then $ABv = BAv = B\lambda v = \lambda Bv$. The first step we use the fact that $AB = BA$, the second step we use the fact that v is an eigenvector of A corresponding to λ, the third step is because λ is a scalar, so it commute with matrix multiplications. So we have $A(Bv) = \lambda \cdot Bv$. On the other hand, $Av = \lambda v \neq 0$, so $v \notin N(A)$. But by condition of the problem, $N(A) = N(B)$, we know $v \notin N(B)$. So $Bv \neq 0$, combined with the equality $A(Bv) = \lambda \cdot Bv$, this means Bv is also a λ -eigenvector for A.

b) First when A has distinct n eigenvalues, the corresponding eigenvectors form a basis of $\mathbb{R}^n$. Secondly when λ_i is non-zero, from part a), we know x_i, Bx_i are both eigenvectors of A corresponding to λ_i. But the eigenspace corresponding to λ_i is 1-dimensional, namely spanned by x_i. So Bx_i is a nonzero multiple of x_i, say $Bx_i = \mu_i \cdot x_i$. So x_i is also an eigenvector of B. When $\lambda_i = 0$, by the condition $N(A) = N(B)$, we know $Bx_i = 0$, so x_i is still an eigenvector of B. B has n linearly independent eigenvectors, so is diagonalizable.