Austin Ford
ford@mit.edu

February 18, 2004

18.06 Linear Algebra
Professor Gilbert Strang

Recitation 8
Bianca Santoro

Problem Set 2

Problem ★ This is about the important $n \times n$ tridiagonal matrices K_n and T_n with $-1, 2, 1$ on all the middle rows. The $(1,1)$ entry of T_n is 1 when the $(1,1)$ entry of K_n is 2:

$$K_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

a) Write MATLAB commands that will create K_5 and T_5 (the `toeplitz` command is useful, or `diag(v,k)` puts a vector v on diagonal k).

b) Factor K_4 and T_4 into the form LDL^T. You can do it by hand, or by using `lu` or `chol` commands.

c) What is L^{-1} for K_4 and T_4?

d) What are the inverse matrices K_4^{-1} and T_4^{-1}?

e) Find a formula (You don't need to prove it, if you don't want.) for the (i,j) entry of T_n^{-1} (the easy one) and K_n^{-1} (a little harder, notice each row of K_4^{-1} and K_5^{-1} on and above the diagonal).

Solution.

a) The following commands will produce K_4.

```
c = [2 -1 0 0];
r = [2 -1 0 0];
K_4 = toeplitz(c,r)
```

The following command will produce T_4.

```
T_4 = [1 -1 0 0; -1 2 -1 0; 0 -1 2 -1; 0 0 -1 2]
```

b) In LDL^T form, we have that

$$K_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (★.1)$$

Similarly, we have

$$T_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{$\bigstar$.2}$$

c) For K_4, we have that $L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{bmatrix}$. For T_4, we have that $L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

d) The inverse of K_4 is $K_4^{-1} = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{6}{5} & \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} & \frac{6}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \end{bmatrix}$. The inverse of T_4 is $T_4^{-1} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

e) A general formula for the (i, j) entry of T_n^{-1} is

$$t_{i,j} = n - \max(i, j) + 1. \tag{$\bigstar$.3}$$

A general formula for the (i, j) entry of K_n^{-1} is

$$k_{i,j} = \frac{\min(i, j)[(n+1) - \max(i, j)]}{n+1}. \tag{$\bigstar$.4}$$

□