

18.06 LINEAR ALGEBRA
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RECITATION 8
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Problem Set 3

Problem ★ Suppose R (an $m \times n$ matrix) is in row reduced echelon form $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ with r nonzero rows and first r pivot columns.

- Describe the column space and nullspace of R .
- Do the same for the $m \times 2n$ matrix $B = \begin{bmatrix} R & R \end{bmatrix}$.
- Do the same for the $2m \times n$ matrix $C = \begin{bmatrix} R \\ R \end{bmatrix}$.
- Finally, do the same for the $2m \times 2n$ matrix $D = \begin{bmatrix} R & R \\ R & R \end{bmatrix}$.

Solution.

- Since R contains the identity as a submatrix, that submatrix contains all of the pivots of R . Therefore, the column space $\mathcal{C}(R) \subset \mathbb{R}^m$ will contain all linear combinations of the first r columns of R . The nullspace $\mathcal{N}(R) \subset \mathbb{R}^n$ will contain all linear combinations of the columns of the matrix $\begin{bmatrix} -F \\ I_{n-r} \end{bmatrix}$, where F is as defined above and I_{n-r} is the $(n-r)$ by $(n-r)$ identity matrix.

- Since B contains the same number of pivot columns as R , we have that $\mathcal{C}(B) = \mathcal{C}(R)$. However, the size of the nullspace has changed, so we now have that

$\mathcal{N}(B) \subset \mathbb{R}^{2n}$ contains all linear combinations of the columns of a matrix $\begin{bmatrix} -F & 0 & I_r \\ I_{n-r} & 0 & 0 \\ 0 & -F & -I_r \\ 0 & I_{n-r} & 0 \end{bmatrix}$.

- In C , we see that the number of rows has changed, so therefore the column space has changed. Since we can eliminate the submatrices below the top ones, we see that C also has r pivot columns; thus, $\mathcal{C}(C)$ contains all linear combinations of the first r columns of C . Also, we can see that the description of the nullspace of C is going to be the same as the description of the nullspace of R since they contain the same structure row-wise. Therefore, $\mathcal{N}(C) = \mathcal{N}(R)$.
- In the case of D , we can see that the row structure is the same as that of B , and the column structure is the same as that of C . Thus, since we can eliminate all of the submatrices below the first row of submatrices, we see that D has r pivot columns

and $\mathcal{C}(D) = \mathcal{C}(C)$. We can also see that all special solutions to systems involving D will also be special solutions to systems involving B , so $\mathcal{N}(D) = \mathcal{N}(B)$.

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