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February 25, 2004

18.06 Linear Algebra
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Recitation 8
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Problem Set 3

Problem ★ Suppose R (an $m \times n$ matrix) is in row reduced echelon form $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ with r nonzero rows and first r pivot columns.

a) Describe the column space and nullspace of R.

b) Do the same for the $m \times 2n$ matrix $B = \begin{bmatrix} R & R \end{bmatrix}$.

c) Do the same for the $2m \times n$ matrix $C = \begin{bmatrix} R \\ R \end{bmatrix}$.

d) Finally, do the same for the $2m \times 2n$ matrix $D = \begin{bmatrix} R & R \\ R & R \end{bmatrix}$.

Solution.

a) Since R contains the identity as a submatrix, that submatrix contains all of the pivots of R. Therefore, the column space $\mathcal{C}(R) \subset \mathbb{R}^m$ will contain all linear combinations of the first r columns of R. The nullspace $\mathcal{N}(R) \subset \mathbb{R}^n$ will contain all linear combinations of the columns of the matrix $\begin{bmatrix} -F \\ I_{n-r} \end{bmatrix}$, where F is as defined above and I_{n-r} is the $(n-r)$ by $(n-r)$ identity matrix.

b) Since B contains the same number of pivot columns as R, we have that $\mathcal{C}(B) = \mathcal{C}(R)$. However, the size of the nullspace has changed, so we now have that $\mathcal{N}(B) \subset \mathbb{R}^{2n}$ contains all linear combinations of the columns of a matrix $\begin{bmatrix} -F & 0 & I_r \\ I_{n-r} & 0 & 0 \\ 0 & -F & -I_r \\ 0 & I_{n-r} & 0 \end{bmatrix}$.

c) In C, we see that the number of rows has changed, so therefore the column space has changed. Since we can eliminate the submatrices below the top ones, we see that C also has r pivot columns; thus, $\mathcal{C}(C)$ contains all linear combinations of the first r columns of C. Also, we can see that the description of the nullspace of C is going to be the same as the description of the nullspace of R since they contain the same structure row-wise. Therefore, $\mathcal{N}(C) = \mathcal{N}(R)$.

d) In the case of D, we can see that the row structure is the same as that of B, and the column structure is the same as that of C. Thus, since we can eliminate all of the submatrices below the first row of submatrices, we see that D has r pivot columns and $\mathcal{C}(D) = \mathcal{C}(C)$. We can also see that all special solutions to systems involving D will also be special solutions to systems involving B, so $\mathcal{N}(D) = \mathcal{N}(B)$.

$\square$