Problem 1

a) The column space is the space of all vectors whose last \( m - r \) coordinates are zero. This is clear since the rank of the matrix \( R \) is \( r \) and the first \( r \) columns of \( R \) are independent.

Denote by \( f_{ij} \) the entry in the \((i, j)\) position in \( F \). The nullspace of \( R \) is the space of all linear combinations of the \( n - r \) vectors

\[
\begin{pmatrix}
-f_{11} & -f_{12} & \cdots & -f_{1(n-r)} \\
-f_{21} & -f_{22} & & \vdots \\
\vdots & \vdots & & \vdots \\
-f_{r1} & -f_{r2} & \cdots & -f_{r(n-r)} \\
1 & 0 & \cdots & 0 \\
0 & 1 & & 0 \\
0 & 0 & & \vdots \\
\vdots & \vdots & & 0 \\
0 & 0 & \cdots & 1
\end{pmatrix}
\]

Clearly these vectors are linearly independent and therefore the dimension of the nullspace is \( n - r \).

b) The column space of the matrix \( B \) is the same as the column space of \( R \).

Denote by \( g_{ij} \) the entry in the \((i, j)\) position in the \( r \times (2n - r) \) matrix \( G := (F \ I \ F) \). Note that we have \( B = \begin{pmatrix} \begin{pmatrix} I & G \\ 0 & 0 \end{pmatrix} \end{pmatrix} \). The nullspace of \( B \) is the space of all linear combinations of the \( 2n - r \) vectors

\[
\begin{pmatrix}
-g_{11} & -g_{12} & \cdots & -g_{1(2n-r)} \\
-g_{21} & -g_{22} & & \vdots \\
\vdots & \vdots & & \vdots \\
-g_{r1} & -g_{r2} & \cdots & -g_{r(2n-r)} \\
1 & 0 & \cdots & 0 \\
0 & 1 & & 0 \\
0 & 0 & & \vdots \\
\vdots & \vdots & & 0 \\
0 & 0 & \cdots & 1
\end{pmatrix}
\]

In terms of the matrix \( F \) we may write the same vectors as
c) The column space of $B$ is the space of vectors in $2m$-dimensional space whose coordinates $b_i$ satisfy the equations

$$b_i = b_{i+m} \quad 1 \leq i \leq m$$
$$b_j = 0 \quad r + 1 \leq j \leq m$$

i.e. they are the vectors of the form

$$\begin{pmatrix} b_1 \\ \vdots \\ b_r \\ 0 \\ \vdots \\ 0 \\ b_1 \\ \vdots \\ b_r \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The nullspace of $C$ is the same as the nullspace of $R$. 

These vectors are clearly linearly independent, and therefore the nullspace of $B$ has dimension $2n - r$. 

\[
\begin{pmatrix}
-f_{11} & -f_{12} & -f_{1(n-r)} & -f_{11} & -f_{11(n-r)} \\
-f_{21} & -f_{22} & -f_{2(n-r)} & -f_{21} & -f_{21(n-r)} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
-f_{r1} & -f_{r2} & -f_{r(n-r)} & -f_{r1} & -f_{r1(n-r)} \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\((n-r)\text{ vectors}\)  \(r\text{ vectors}\)  \((n-r)\text{ vectors}\)
d) The column space of $D$ is the same as the column space of $C$. The nullspace of $D$ is the same as the nullspace of $B$.

**Problem 2**

a) The nullspace of $A$ is contained in the nullspace of $A^2$. The reason is that if $Ax = 0$, i.e. if $x$ is in the nullspace of $A$, then $A^2x = A \cdot (Ax) = 0$. Thus $x$ is also in the nullspace of $A^2$. Similarly we have

$$N(A) \subset N(A^2) \subset N(A^3) \subset \ldots$$

Note that one can prove that if $A$ is an $n \times n$ matrix, then one has $N(A^n) = N(A^{n+1}) = \ldots$

b) The nullspace is by definition the set of all vectors $v$ such that $\frac{d^2}{dx^2} v = 0$. This means that the polynomial $v$ must be linear: $v = cx + d$. Thus the nullspace is the space of polynomials of degree at most one.

The nullspace of $\left( \frac{d^2}{dx^2} \right)^2$ is the nullspace of the composition of $\frac{d^2}{dx^2}$ with itself: it is the nullspace of $\frac{d^4}{dx^4}$. Thus the nullspace of $\frac{d^4}{dx^4}$ is the space of all polynomials of degree at most three: $v = ax^3 + bx^2 + cx + d$. 

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