Exam 2, Friday April 1st, 2005

Solutions

Question 1. The vector \(a_1\) can be any non-zero positive multiple of \(q_1\). The vector \(a_2\) can be any multiple of \(q_1\) plus any non-zero positive multiple of \(q_2\):

\[
a_1 = cq_1 \quad a_2 = c_1q_1 + c_2q_2
\]

with \(c, c_1 > 0\).

Question 2. We want to find the least squares solution to the equation

\[ax = b\]

and we know that it is enough to multiply both sides by \(a^T\) and solve the resulting system:

\[a^T ax = a^T b \quad \implies \quad x = \frac{a \cdot b}{a \cdot a}\]

Question 3. The vectors \((-1, 1, 0)^T\) and \((-1, 0, 1)^T\) form a basis for the subspace \(x + y + z = 0\). Let \(A\) be the matrix whose columns are the two vectors found above. Thus the projection matrix \(P\) onto the subspace \(x + y + z = 0\) is

\[
P = A (A^T A)^{-1} A^T = \begin{pmatrix}
-1 & -1 \\
1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
2 & 1 \\
1 & 2
\end{pmatrix}^{-1} \begin{pmatrix}
-1 & 1 & 0 \\
-1 & 0 & 1
\end{pmatrix} = \frac{1}{3} \begin{pmatrix}
-1 & -1 \\
1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
-1 & 2 & -1 \\
-1 & -1 & 2
\end{pmatrix} = \frac{1}{3} \begin{pmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2
\end{pmatrix}
\]

The projection of \((1, 2, 6)^T\) onto the plane \(x + y + z = 0\) is thus simply

\[
p = P \begin{pmatrix}
1 \\
2 \\
6
\end{pmatrix} = \begin{pmatrix}
-2 \\
-1 \\
3
\end{pmatrix}
\]

Question 4. Looking at the first row of \(A\) we deduce that

\[
det A = det \begin{pmatrix}
1 & 1 & 0 \\
1 & 2 & 3 \\
1 & 3 & 9
\end{pmatrix} = 9 - 6 = 3
\]
Of course, \( \det(A^{-1}) = \frac{1}{3} \). Finally

\[
(A^{-1})_{12} = \frac{-C_{21}}{\det A} = \frac{-1}{3} \det \begin{pmatrix} 0 & 1 & 0 \\
2 & 1 & 3 \\
3 & 1 & 9 \\
\end{pmatrix} = \frac{-(9)}{3} = 3
\]

**Question 5.** (a) The column space of \( QQ^T \) is at most two dimensional, since the matrix \( QQ^T \) is \( 4 \times 4 \), it cannot have rank four. Thus \( \det QQ^T = 0 \).

Similarly, the matrix \( Q Q \) has dependent columns, and therefore \( \det[ Q Q ] = 0 \).

(b) Using the projection formula,

\[
p = Q(Q^TQ)^{-1}Q^Tb = QIQ^Tb = QQ^Tb
\]

(c) The error vector \( e = b - p \) is contained in the left null-space of \( Q \), the nullspace of \( Q^T \). To check this, we compute

\[
Q^Te = Q^T (b - QQ^Tb) = Q^Tb - Q^TQQ^Tb = Q^Tb - IQ^Tb = 0
\]

**Question 6.** The product \( P_2P_1 \) is projection onto the column space of \( P_1 \), followed by the projection onto the column space of \( P_2 \). Since the column space of \( P_2 \) contains the column space of \( P_1 \), the second projection does not change the vectors anymore. Thus

\[
P_2P_1 = P_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{6} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}
\]