18.06 Problem Set 6 - Solutions
Due Wednesday, April 11, 2007 at 4:00 p.m. in 2-106

**Problem 1** Wednesday 4/4
Do problem 9 of section 6.1 in your book.

**Solution 1**

(a) Multiply $A$ on the left to both sides of the equation $Ax = \lambda x$ to get $AAx = A\lambda x$. But $AAx = A^2x$ and $A\lambda x = \lambda Ax = \lambda^2 x$, so we have $A^2x = \lambda^2 x$, which means that $\lambda^2$ is an eigenvalue of $A^2$.

(b) Multiply $\lambda^{-1}A^{-1}$ on the left to both sides of the equation $Ax = \lambda x$ to get $\lambda^{-1}A^{-1}Ax = \lambda^{-1}A^{-1}\lambda x$. But $\lambda^{-1}A^{-1}Ax = \lambda^{-1}x$ and $\lambda^{-1}A^{-1}\lambda x = A^{-1}\lambda^2 = A^{-1}x$, so we have $A^{-1}x = \lambda^{-1}x$, which means that $\lambda^{-1}$ is an eigenvalue of $A^{-1}$.

(c) Add $x$ to both sides of the equation $Ax = \lambda x$ to get $Ax + x = \lambda x + x$. But this is exactly $(A + I)x = (\lambda + 1)x$, which means that $\lambda + 1$ is an eigenvalue of $A + I$.

**Problem 2** Wednesday 4/4
Do problem 28 of section 6.1 in your book.

**Solution 2**

The matrix $A$ has rank 1 (all rows are equal), which implies that 0 is an eigenvalue of $A$ (the three independent vectors in the nullspace of $A$ are the three independent eigenvectors with eigenvalue 0). Now let us find other eigenvalues. If $(x, y, z, w)^T$ is an eigenvector with eigenvalue $\lambda \neq 0$, then:

$$A \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x + y + z + w \\ x + y + z + w \\ x + y + z + w \\ x + y + z + w \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

But this implies that $x = y = z = w$ and furthermore $\lambda = 4$. Thus, the four eigenvalues of $A$ are 0, 0, 0, 4.

The matrix $B$ has rank 2 (rows 1 and 3 are equal, rows 2 and 4 are equal), which implies that 0 is an eigenvalue of $A$ (the two independent vectors in the nullspace of $A$ are the two independent eigenvectors with eigenvalue 0). Now let us find other eigenvalues. If $(x, y, z, w)^T$ is an eigenvector with eigenvalue $\lambda \neq 0$, then:

$$A \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x + z \\ y + w \\ x + z \\ y + w \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

But this implies that $x = z$ and $y = w$, and furthermore $\lambda = 2$ (we get two independent eigenvectors here: $(1, 0, 1, 0)^T$ and $(0, 1, 0, 1)^T$). Thus, the four eigenvalues of $A$ are 0, 0, 2, 2.

**Problem 3** Wednesday 4/4
Do problem 33 of section 6.1 in your book.
Solution 3

(a) Since \( u, v, w \) are independent, any vector \( x \) can be written as a linear combination of those, \( x = c_1 u + c_2 v + c_3 w \). Then

\[
Ax = A(c_1 u + c_2 v + c_3 w) = c_1 Au + c_2 Av + c_3 Aw = 3c_2 v + 5c_3 w
\]

If \( Ax = 0 \), then we must have \( c_2, c_3 = 0 \), so the vectors in the nullspace of \( A \) are multiples of \( u \), and a basis for \( N(A) \) is the vector \( u \).

All vectors \( Ax \) in the column space of \( A \) are linear combinations of \( v \) and \( w \): a basis for \( \text{C}(A) \) consists of the vectors \( v \) and \( w \).

(b) We want to find the solutions of \( Ax = v + w \). Let \( x = c_1 u + c_2 v + c_3 w \). Then as seen above \( Ax = 3c_2 v + 5c_3 w \), so we must have \( c_2 = \frac{1}{3} \) and \( c_3 = \frac{1}{5} \), while \( c_1 \) can take any values. The solution for this is of the form \( x = c_1 u + \frac{1}{3} v + \frac{1}{5} w \).

(c) \( Ax = u \) has no solution because if it did then \( u \) would be in the column space.

Problem 4 Wednesday 4/4

Let \( A \) be a fixed \( n \times n \) matrix. We would like to find a matrix \( B \) such that \( AB = BA \). This is the same as solving \( AB - BA = 0 \) matrix. It turns out that this is a system of \( n^2 \) equations on the entries of \( B \) (which are unknown). Since all these equations are linear, we can associate this system to a matrix \( M \). Find an eigenvector of this matrix \( M \) with its corresponding eigenvalue.

Solution 4

We have \( Mx = 0 \) exactly when the vector \( x \) corresponds to a matrix \( B \) that satisfies \( AB - BA = 0 \). But there is one case of such a matrix that is quite simple: just take \( B \) to be the matrix \( A \) itself! Then clearly \( AA - AA = 0! \) So if \( x \) is the vector corresponding to the matrix \( A \), then \( Mx = 0 \), and this means that \( x \) is an eigenvector of \( M \), with eigenvalue 0.

Problem 5 Monday 4/9

Do problem 7 of section 6.2 in your book.

Solution 5

We begin by computing the eigenvalues of \( A \), solving \( \det(A - \lambda I) = 0 \) for \( \lambda \).

\[
\det(A - \lambda I) = \det \begin{bmatrix} 4 - \lambda & 0 \\ 1 & 2 - \lambda \end{bmatrix} = (4 - \lambda)(2 - \lambda)
\]

The eigenvalues are \( \lambda = 2 \) and \( \lambda = 4 \).

Now, for each eigenvalue \( \lambda \), we want to find the eigenvectors, i.e., vectors in the nullspace of \( A - \lambda I \).

For \( \lambda = 2 \), we have \( A - 2I = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} \), so \( N(A - 2I) \) is generated by the vector \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \). Thus, any vector of the form \( \begin{bmatrix} 0 \\ a \end{bmatrix} \) with \( a \neq 0 \) is a suitable eigenvector. For \( \lambda = 4 \), we have \( A - 4I = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix} \), so \( N(A - 4I) \) is generated by the vector \( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \). Thus, any vector of the form \( \begin{bmatrix} 2b \\ b \end{bmatrix} \) with \( b \neq 0 \) is a suitable eigenvector. Writing in these vectors as columns of a matrix we get a matrix \( S \) that diagonalizes \( A \):

\[
S = \begin{bmatrix} 0 & 2b \\ a & b \end{bmatrix} \quad \Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}
\]

If we switch the columns, we still get a matrix that diagonalizes \( A \):

\[
S = \begin{bmatrix} 2b & 0 \\ b & a \end{bmatrix} \quad \Lambda = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}
\]
We know that if \( x \) is an eigenvector of \( A \) (with eigenvalue \( \lambda \)), then it is also an eigenvector of \( A^{-1} \) (with eigenvalue \( \lambda^{-1} \)), so the same matrices \( S \) work for diagonalizing \( A^{-1} \) (the diagonal matrix changes accordingly).

**Problem 6 Monday 4/9**

Do problem 10 of section 6.2 in your book.

**Solution 6**

The equations \( G_{k+2} = \frac{1}{2}G_{k+1} + \frac{1}{2}G_k \) and \( G_{k+1} = G_{k+1} \) can be written in matrix form as

\[
\begin{bmatrix}
G_{k+2} \\
G_{k+1}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & 0
\end{bmatrix}
\begin{bmatrix}
G_{k+1} \\
G_k
\end{bmatrix}
\]

(a) Firstly, we find the eigenvalues of \( A = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & 0
\end{bmatrix} \) by solving \( \det(A - \lambda I) = 0 \) for \( \lambda \):

\[
\det(A - \lambda I) = \det\begin{bmatrix}
\frac{1}{2} - \lambda & \frac{1}{2} \\
\frac{1}{2} & -\lambda
\end{bmatrix} = (\lambda - 1)(\lambda + \frac{1}{2})
\]

The eigenvalues are \( \lambda = 1 \) and \( \lambda = -\frac{1}{2} \).

Now, we find the eigenvectors for each \( \lambda \). For \( \lambda = 1 \), we have \( A - I = \begin{bmatrix}
-\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -1
\end{bmatrix} \), so \( N(A - I) \) is generated by the vector \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \), and this is an eigenvector. For \( \lambda = -\frac{1}{2} \), we have \( A + \frac{1}{2} I = \begin{bmatrix}
1 & \frac{1}{2} \\
1 & \frac{1}{2}
\end{bmatrix} \), so \( N(A + \frac{1}{2} I) \) is generated by the vector \( \begin{bmatrix} -1 \\ 2 \end{bmatrix} \), and this is another eigenvector.

(b) The eigenvector matrix is \( S = \begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix} \), its inverse is \( S^{-1} = \frac{2}{3} \begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix} \), and the eigenvalue matrix is \( \Lambda = \begin{bmatrix}
1 & 0 \\
0 & -\frac{1}{2}
\end{bmatrix} \). Then \( A^n = S\Lambda^nS^{-1} \). As \( n \to \infty \),

\[
\Lambda^n = \begin{bmatrix}
1 & 0 \\
0 & -\frac{1}{2}
\end{bmatrix}^n = \begin{bmatrix}
1^n & 0 \\
0 & (-\frac{1}{2})^n
\end{bmatrix} \to \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}
\]

Then,

\[
A^n = S\Lambda^nS^{-1} \to \begin{bmatrix}
1 & -1 \\
1 & 2
\end{bmatrix} \begin{bmatrix}
1^n & 0 \\
0 & (-\frac{1}{2})^n
\end{bmatrix} \begin{bmatrix}
2 & 1 \\
-1 & 1
\end{bmatrix} = \begin{bmatrix}
\frac{2}{3} & 2 \\
\frac{1}{3} & 1
\end{bmatrix}
\]

(c) Applying \( A \) repeatedly to \( \begin{bmatrix}
G_1 \\
G_0
\end{bmatrix} \) we get

\[
\begin{bmatrix}
G_{n+1} \\
G_n
\end{bmatrix} = A^n \begin{bmatrix}
G_1 \\
G_0
\end{bmatrix}
\]

But \( A^n \begin{bmatrix}
G_1 \\
G_0
\end{bmatrix} \to \frac{1}{3} \begin{bmatrix}
2 & 1 \\
2 & 1
\end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix}
\frac{2}{3} \\
\frac{2}{3}
\end{bmatrix} \), which implies that \( \begin{bmatrix}
G_{n+1} \\
G_n
\end{bmatrix} \to \begin{bmatrix}
\frac{2}{3} \\
\frac{2}{3}
\end{bmatrix} \), that is, the Gibonacci numbers \( G_n \) approach \( \frac{2}{3} \).

**Problem 7 Monday 4/9**

Do problems 15 and 16 of section 6.2 in your book.

**Solution 7**

Problem 15

If the eigenvalues of \( A \) are 2, 2, 5 then the matrix is certainly invertible, as its determinant is \( \det A = 2 \times 2 \times 5 = 20 \neq 0 \). Such a matrix could be diagonalizable or not, depending on whether or not there are two independent eigenvectors for the eigenvalue 2.
Problem 16
If the only eigenvectors of \( A \) are multiples of \((1, 4)\), i.e., there is only one independent eigenvector, then \( A \) must have a repeated eigenvalue, as eigenvectors corresponding to distinct eigenvalues are independent. This matrix is not diagonalizable, since there aren’t enough independent eigenvectors (we needed two of them for this 2-by-2 matrix). As for \( A \) being invertible or not, it depends on this repeated eigenvalue being zero: \( \det A = \lambda^2 = 0 \) iff \( \lambda = 0 \).

Problem 8 Monday 4/9
Do problem 22 of section 6.2 in your book.

Solution 8
We begin by computing the eigenvalues of \( A \) by solving \( \det(A - \lambda I) = 0 \) for \( \lambda \):
\[
\det(A - \lambda I) = \det \begin{bmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{bmatrix} = (2 - \lambda)^2 - 1 = (1 - \lambda)(3 - \lambda)
\]
The eigenvalues are \( \lambda = 1 \) and \( \lambda = 3 \). Now, we find the corresponding eigenvectors. For \( \lambda = 1 \), we have \( A - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \), so \( N(A - I) \) is generated by the vector \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \), which is an eigenvector of \( A \). For \( \lambda = 3 \), we have \( A - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \), so \( N(A - 3I) \) is generated by the vector \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \), which is another eigenvector of \( A \). The eigenvector matrix is
\[
S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},
\]
its inverse is
\[
S^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},
\]
and the corresponding diagonal matrix is
\[
\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.
\]
We have \( A = SAS^{-1} \), and so \( A^k = SAS^{-1} \):
\[
A^k = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3^k + 1 & 3^k - 1 \\ 3^k - 1 & 3^k + 1 \end{bmatrix}
\]

Problem 9 Monday 4/9
Do problem 28 of section 6.2 in your book.

Solution 9
Let \( S \) be the set of 4-by-4 matrices that are diagonalized by the same eigenvector matrix \( S \), i.e., matrices \( A \) such that \( S^{-1}AS \) is a diagonal matrix. We want to prove that this is a subspace:
Suppose \( A \in S \), with \( S^{-1}AS = \Lambda \) diagonal matrix, and let \( c \) be a scalar. Then,
\[
S^{-1}(cA)S = cS^{-1}AS = c\Lambda
\]
is also a diagonal matrix. Thus, \( cA \) is diagonalized by \( S \), and \( cA \in S \).
Suppose \( A_1, A_2 \in S \), with \( S^{-1}A_1S = \Lambda_1 \) and \( S^{-1}A_2S = \Lambda_2 \) diagonal matrices. Then,
\[
S^{-1}(A_1 + A_2)S = S^{-1}A_1S + S^{-1}A_2S = \Lambda_1 + \Lambda_2
\]
is also a diagonal matrix (the sum of two diagonal matrices is diagonal). Thus, $A_1 + A_2$ is diagonalized by $S$, and $A_1 + A_2 \in S$.

Alternatively, let $v_1, v_2, \ldots, v_n$ be the column vectors of $S$. Then $S$ is the set of 4-by-4 matrices that have $v_1, v_2, \ldots, v_n$ as eigenvectors. But the eigenvectors of $cA$ are the same as those of $A$ (prove this!), and if $A_1, A_2$ have the same eigenvectors, then so does $A_1 + A_2$ (prove this!).

In the case that $S$ is the identity matrix, then $S^{-1}AS = I^{-1}AI = A$ must be a diagonal matrix. Thus, $S$ is the space of 4-by-4 diagonal matrices, which has dimension 4.

**Problem 10 Monday 4/9**

(a) Give an example of a $3 \times 3$ matrix $A \neq 0$ such that $A^2 \neq 0$ but $A^3 = 0$. Find your $A$ find all the eigenvalues and the eigenvectors.

(b) Now, let $B$ be a diagonalizable matrix such that there exists some positive integer $k$ such that $B^k = 0$. Prove that $B = 0$.

(c) Does part (b) contradict part (a)? Explain your answer.

**Solution 10**

(a) One such example is $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Then, $A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. To find the eigenvalues we solve $\det(A - \lambda I) = 0$ for $\lambda$.

$$
\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{bmatrix} = -\lambda^3
$$

so $\lambda = 0$ is the only eigenvalue. There is only one eigenvector, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, which spans the nullspace of $A - 0I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) Now, let $B$ be a diagonalizable matrix such that $B^k = 0$ for some $k$. Since $B$ is diagonalizable, we can write $\Lambda^k = S^{-1}B^kS = S^{-1}0S = 0$. But because $\Lambda$ is a diagonal matrix, this implies that $\Lambda = 0$:

$$
\Lambda^k = \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} = 0 \implies \forall i \lambda_i^k = 0 \implies \forall i \lambda_i = 0 \implies \Lambda = 0
$$

(c) No, there is no contradiction, because $A$ in (a) was not diagonalizable (not enough independent eigenvectors)!

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