Problem 1: Do problem 7 from section 2.7 (pg. 105) in the book.

Solution (2+3+3+2 points)

a) False. One example is when \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \).

b) False. We check by taking the transpose:

\[
(AB)^T = B^T A^T = BA
\]

So, any time we have symmetric matrices \( A \) and \( B \) which give different answers when we multiply them in different orders, the hypothesis will fail. One example is \( A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \), so that

\[
AB = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}
\]

is not symmetric.

c) True. If any matrix \( B \) is symmetric, then its inverse is also symmetric. So, if \( A^{-1} \) were symmetric, then \( A \) would need to be symmetric as well.

d) True. We check by taking the transpose:

\[
(ABC)^T = C^T B^T A^T = CBA
\]

Problem 2: Do problem 10 from section 3.1 (pg. 119). (Give explanations of just a sentence or two.)

Solution (10 points)

a) This is a subspace; it is the nullspace of the matrix \( A = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \).

b) This is not a subspace, as it does not contain the 0 vector.
c) This is not a subspace; for example, the vectors

\[
v = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad w = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}
\]

both have \( b_1b_2b_3 = 0 \), but their sum does not.

d) This is a subspace. Any time we take all linear combinations of a set of vectors, we get a subspace.

e) This is a subspace; it is the nullspace of the matrix \( A = [1 \ 1 \ 1] \).

f) This is not a subspace. If we take the vector

\[
v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}
\]

and multiply it by the scalar \(-1\), it is no longer is in the set.

**Problem 3:** Consider the system of equations

\[
\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 5 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
\]

a) For which right sides (find a condition on \( b_1, b_2, b_3 \)) is this system solvable?

b) Call the coefficient matrix \( A \). Is the vector \((2, 5, -2)\) in the column space of \( A \)? How about \((1, 2, 3)\)?

c) Suppose we add a fourth column \((2, 5, -2)\) to \( A \). How does the column space change? What if we added the column \((1, 2, 3)\) instead?

**Solution** (5 points)

a) We reduce the augmented matrix:

\[
\begin{bmatrix} 1 & 4 & 2 & b_1 \\ 2 & 8 & 5 & b_2 \\ -1 & -4 & -2 & b_3 \end{bmatrix} \leadsto \begin{bmatrix} 1 & 4 & 2 & b_1 \\ 0 & 0 & 1 & b_2 - 2b_1 \\ -1 & -4 & -2 & b_3 \end{bmatrix} \leadsto \begin{bmatrix} 1 & 4 & 2 & b_1 \\ 0 & 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 + b_1 \end{bmatrix}
\]
The conditions come from the 0 rows. In this case, there is only one 0 row, giving us the condition \( b_1 + b_3 = 0 \).

b) The vector \((2, 5, -2)\) is in the column space, because \( b_1 + b_3 = 0 \). The second vector is not.

c) If we append a column to \( A \) that is already in the column space, it doesn’t change the column space. That is, we get no new linear combinations by adding in a vector already in the space. Of course, if the column is not in \( C(A) \), then the space gets larger.

In this case, adding \((2, 5, -2)\) doesn’t change anything, while adding \((1, 2, 3)\) makes the column space equal to all of \( \mathbb{R}^3 \).

**Problem 4:** Do problem 9 from section 3.2 (pg. 131).

<table>
<thead>
<tr>
<th>Solution (5 points)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) False. A counter example is a square 0 matrix (every variable is free!).</td>
</tr>
<tr>
<td>b) True. Every column is a pivot column, so there are no free variables.</td>
</tr>
<tr>
<td>c) True. There can be at most one pivot per column (because we always eliminate so that there are 0s below each pivot).</td>
</tr>
<tr>
<td>d) True. There can be at most one pivot per row (by definition).</td>
</tr>
</tbody>
</table>

**Problem 5:** Do problem 20 from section 3.2 (pg. 132).

<table>
<thead>
<tr>
<th>Solution (10 points)</th>
</tr>
</thead>
</table>
| We have that \( A \) is a 4 by 5 matrix, with 4 pivots. This means that only one column can not have a pivot. Since \((\text{col } 1)+(\text{col } 3)+(\text{col } 5) = 0\), we know that \( \text{col } 5 \) can not have a pivot. Thus, it must be the only one without a pivot! This means that the fifth variable is free, but the others are not.

Since there is only one free variable, there is one special solution, and the nullspace is one-dimensional (it is exactly the scalar multiples of the special solution). The column equation tells us that the vector

\[
[1] \\
0 \\
1 \\
0 \\
1
\]

3
is in the nullspace. It is also the special solution, since $x_5 = 1$.

**Problem 6:** Do problem 21 from section 3.2 (pg. 132). (Make sure the matrix you construct has exactly the nullspace asked for in the problem, and no larger.)

**Solution** (10 points)

There are several ways to think about it. One way is as follows. Note that if $A$ is a matrix with these two vectors in its nullspace, this means that the dot product of the first row of $A$ with either of these vectors is 0. Switching the roles, we see that row 1 of $A$ must be in the nullspace of

$$B = \begin{bmatrix} 2 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix}$$

Of course, the same argument shows that every row of $A$ must be in the nullspace of $B$.

So, we find the nullspace of $B$:

$$\begin{bmatrix} 2 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & -2 & -\frac{3}{2} & 1 \end{bmatrix}$$

Rescaling and putting into RR echelon form, we end up with

$$\begin{bmatrix} 1 & 0 & -\frac{1}{4} & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} & -\frac{1}{2} \end{bmatrix}$$

So, the special solutions are

$$v = \begin{bmatrix} 1/4 \\ -3/4 \\ 1 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix}$$

Each row of $A$ must be a linear combination of $v$ and $w$.

Now, since $A$ has 4 columns and exactly two independent vectors in its nullspace, $A$ must have rank 2. Thus, $A$ must have at least two rows, consisting of linearly independent combinations of $v$ and $w$. Some examples would be

$$\begin{bmatrix} 1/4 & -3/4 & 1 & 0 \\ -1/2 & 1/2 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1/2 & -3/2 & 2 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
Problem 7: a) Do problem 1 from section 3.3 (pg. 141).
    b) Do problem 3 in section 3.3 (pg. 141).

Solution (5+5 points)
    a) Definitions a and c are correct. Definition b is not: the identity matrix has full rank, but b gives the answer 0. Definition d is also not correct: for example the matrix

\[ A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \]

is in row reduced echelon form and has rank 2, but has 4 ones.

    b) For the original \( A \), we obtain

\[ R = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \]

For the block matrix \( B \), the RR echelon form is the block matrix \( \begin{bmatrix} R & R \end{bmatrix} \). We can see this because the row operations that reduce the left hand side \( A \) to row reduced form will also reduce the right hand side.

The block matrix \( C \) will give us the matrix

\[ \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

Problem 8: Do problem 17 in section 3.3 (pg. 143).

Solution (7+3 points)
    a) Suppose we can write \((\text{col } j)\) of \( B \) in an equation

\[ (\text{col } j) = \sum_{i<j} a_i (\text{col } i) \]

We can multiply both sides of this equation on the left by \( A \). Since matrix multiplication distributes over addition, we find that

\[ A(\text{col } j) = \sum_{i<j} a_i A(\text{col } i) \]
Of course, these are just the columns of $AB$. This shows that every free column of $B$ is still a free column of $AB$, i.e. $rk(AB) \leq rk(B)$. Note that left multiplying by $A$ can actually introduce new relations between the columns, so that the rank can go down. We’ll see an example in part b.

b) We have $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. If $A_1$ is the identity matrix, then $rk(A_1B) = rk(B) = 1$. If $A_2$ is the 0 matrix, then $rk(A_2B) = rk(0) = 0$.

**Problem 9:** Define the matrix

$$A = \begin{bmatrix}
1 & 2 & 2 & 4 & 6 \\
1 & 2 & 3 & 6 & 9 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 1 & 1 & 0
\end{bmatrix}$$

a) Reduce $A$ to ordinary echelon form. What are the pivots? What are the free variables?

b) Find a special solution for each free variable. (Set the free variable to 1. Set the other variables to 0.)

c) By combining the special solutions, describe every solution to $Ax = 0$.

d) What is the rank of $A$? Which columns will generate the column space $C(A)$?

**Solution** (3+2+2+3 points)

a) We reduce:

$$\begin{bmatrix}
1 & 2 & 2 & 4 & 6 \\
1 & 2 & 3 & 6 & 9 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 1 & 1 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 2 & 2 & 4 & 6 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 1 & 1 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 2 & 2 & 4 & 6 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 2 & 2 & 4 & 6 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 2 & 2 & 4 & 6 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & -1 & -3 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Columns 1,3,4 are pivots, and columns 2,5 are free.
b) The special solutions are:
\(x_2 = 1, x_5 = 0\) leads to \((-2, 1, 0, 0, 0)\).
\(x_2 = 0, x_5 = 1\) leads to \((0, 0, 3, -3, 1)\).

c) The nullspace is all linear combinations of the special solutions.

d) The rank of \(A\) is 3 (the number of pivots), and these columns (1,3,4) generate the column space.

**Problem 10:** Do problem 4 in section 3.4 (pg. 152).

**Solution** (10 points)
We first find a particular solution by reducing the augmented matrix:

\[
\begin{bmatrix}
1 & 3 & 1 & 2 & 1 \\
2 & 6 & 4 & 8 & 3 \\
0 & 0 & 2 & 4 & 1
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & 3 & 1 & 2 & 1 \\
0 & 0 & 2 & 4 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The reduction of \(A\) has two pivots (columns 1 and 3) and two free variables (columns 2 and 4). To find a particular solution, we set the free variables to 0 and solve \(Ux = (1, 1, 0)\). Setting \(x_2 = x_4 = 0\) implies \(x_1 = 1/2\) and \(x_3 = 1/2\). (Of course there are other particular solutions; we just need to find one by setting the free variables to whatever we like. 0 is an easy choice.)

Now, we find the nullspace of \(A\) using special solutions. We’ve already done the work to reduce \(A\), so we forget about the augmented part and solve \(Ux = 0\). We find:

\(x_2 = 1, x_4 = 0\) implies \(x_1 = -3\) and \(x_3 = 0\).
\(x_2 = 0, x_4 = 1\) implies \(x_1 = 0\) and \(x_3 = -2\).

The complete solution is the particular solution plus all the vectors in the nullspace:

\[
x = x_p + x_n = \begin{bmatrix}
1/2 \\
0 \\
0 \\
1/2 \\
0
\end{bmatrix} + c_1 \begin{bmatrix}
-3 \\
1 \\
0 \\
0 \\
0
\end{bmatrix} + c_2 \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
-2
\end{bmatrix}
\]

for any choice of \(c_1\) and \(c_2\).
Problem 11: Do problem 21 in section 3.4 (pg. 154).

Solution (5+5 points)

a) Here \( A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \). It has one pivot and two free variables. We find a particular solution by setting the free variables to 0 and solving \( Ux = 4 \): we get

\[
x_p = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}
\]

The special solutions are found by setting one free variable to 1, one to 0, and solving \( Ux = 0 \). We find special solutions

\[
\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}
\]

Thus

\[
x_p + x_n = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}
\]

for any choice of \( c_1 \) and \( c_2 \).

b) We augment and reduce to get

\[
\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & -2 & 0 & 0 \end{bmatrix}
\]

So, a particular solution is found by setting \( x_3 = 0 \) and solving \( Ux = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \), thus \( x_1 = 4 \) and \( x_2 = 0 \). The nullspace is found by setting the free variable to 1 and solving \( Ux = 0 \). In sum we get

\[
x_p + x_n = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

for any choice of \( c_1 \).