Problem 1: Do problem 1 in section 6.3 (pg. 315) in the book.

Solution (10 points)
We solve a linear system of differential equations by taking
\[ u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 \]
where \( \lambda_1, \lambda_2 \) are the eigenvalues, \( x_1, x_2 \) are the eigenvectors, and \( c_1, c_2 \) are constants that satisfy \( c_1 x_1 + c_2 x_2 = u(0) \).

To write down the matrix exponential explicitly, we must find the eigenvalues and eigenvectors of \( A \). Since this \( A \) is diagonal, its eigenvalues are just the diagonal entries, i.e. \( \lambda_1 = 4 \) and \( \lambda_2 = 1 \). The eigenvectors are \( x_1 = (1, 0) \) and \( x_2 = (1, -1) \). Finally, if \( u(0) = (5, -2) \) we must find how to write \( u(0) \) as a linear combination of \( x_1 \) and \( x_2 \). We do this by solving the equation
\[
\begin{bmatrix}
1 & 1 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= 
\begin{bmatrix}
5 \\
-2
\end{bmatrix}
\]
We get \( c_2 = 2 \) and \( c_1 = 3 \). So, the final equation is
\[ u(t) = 3e^{4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} \]

Problem 2: Do problem 3 in section 6.3 (pg. 315).

Solution (10 points)
To linearize this system, we identify \( u \) with the vector \( [y, y']^T \), so that we have two equations \( dy/dt = y' \) and \( dy'/dt = y'' = 4y + 5y' \). That is, we can “decouple” the differential equation by adding \( y' \) as a new variable, to obtain the system
\[
\begin{bmatrix}
\frac{dy}{dt} \\
\frac{dy'}{dt}
\end{bmatrix}
= 
\begin{bmatrix}
0 & 1 \\
4 & 5
\end{bmatrix}
\begin{bmatrix}
y \\
y'
\end{bmatrix}
\]
We call the coefficient matrix \( A \) as usual. The eigenvalues of \( A \) satisfy the equation \( \lambda^2 - 5\lambda - 4 = 0 \), so the eigenvalues are \( \lambda_1 = \frac{1}{2}(5 + \sqrt{41}) \) and \( \lambda_2 = \frac{1}{2}(5 - \sqrt{41}) \).
Another way to find the eigenvalues is to substitute \( y = e^{\lambda t} \) into the differential equation. We obtain

\[
\lambda^2 e^{\lambda t} = 5\lambda e^{\lambda t} + 4e^{\lambda t}
\]

Dividing by \( e^{\lambda t} \), we find the same relationship \( \lambda^2 - 5\lambda - 4 = 0 \).

**Problem 3:**

a) Do problem 17 in section 6.3 (pg. 317).

b) Do problem 24 in section 6.3 (pg. 318).

**Solution** (5+5 points)

a) The infinite series for \( e^{Bt} \) is

\[
e^{Bt} = I + tB + \frac{1}{2}t^2B^2 + \ldots
\]

However, since \( B^2 = 0 \), all the terms of this sequence will be zero except for the first two. Thus

\[
e^{Bt} = I + tB = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}
\]

The derivative is

\[
d(e^{Bt})/dt = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}
\]

Of course, this is the same thing as \( Be^{Bt} \) (just multiply it out).

b) First, recall that \( e^{A+B} = e^Ae^B \) whenever \( AB = BA \). The matrices \( A \) and \( -A \) always commute (both products are \( -A^2 \)), so \( e^{At}e^{-At} = e^0 = I \). Thus \( e^{At} \) is always invertible. You could also check this by multiplying out the power series formally.

Second, we know that \( e^{At} \) has diagonalization \( Se^{At}S^{-1} \). That is, the eigenvalues of \( e^{At} \) are just \( e^{\lambda t} \) for eigenvalues \( \lambda \) of \( A \). However, \( e^{At} \) is never 0, so \( e^{At} \) never has 0 for an eigenvalue, meaning that it is always invertible.

**Problem 4:**

a) Do problem 4 in section 6.4 (pg. 327).

b) Do problem 10 in section 6.4 (pg. 327).

**Solution** (5+5 points)

a) We need to diagonalize \( A \); since \( A \) is symmetric, we know that we will be able to pick perpendicular eigenvectors. If we normalize these eigenvectors to length 1, the eigenvector matrix will be orthogonal. \( A \) has eigenvalues given by the equation
\[ \lambda^2 - 5\lambda - 50 = 0, \] so \( A \) has eigenvalues \( \lambda_1 = 10 \) and \( \lambda_2 = -5 \). The corresponding eigenvectors of unit length are \( x_1 = \frac{1}{\sqrt{5}}[1, 2]^T \) and \( x_2 = \frac{1}{\sqrt{5}}[-2, 1]^T \). So

\[
Q = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}
\]

b) The flaw here is that \( x^Tx \) is not necessarily a real number (and neither is \( x^TAx \)). We know that \( x^Hx \) is always real, since it is the length of \( x \) squared. But in general \( x^Tx \) is not real - take for example the one-component vector \( x = [1 + i] \).

**Problem 5:** Do problem 15 in section 6.4 (pg. 328).

**Solution** (10 points)

We diagonalize

\[
A = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}
\]

The eigenvalues are given by the equation \( \lambda^2 = 0 \), so the only eigenvalue is \( \lambda = 0 \). The eigenvectors are then given by the nullspace of \( A \). We find this using row reduction, just as for real matrices:

\[
\begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \sim \begin{bmatrix} i & 1 \\ 1 & 0 \end{bmatrix}
\]

The nullspace is one dimensional, meaning that every eigenvector is a multiple of \([1, -i]^T\).

**Problem 6:** Do problem 24 in section 6.4 (pg. 329).

**Solution** (10 points) (No justifications necessary.)

We start with \( A \). It is definitely invertible. It is orthogonal since \( P^T = P^{-1} \) (both are equal to \( P \)). It is not a projection matrix because \( P^2 \neq P \). It is clearly a permutation matrix. It is diagonalizable because it is symmetric (so that we can find a basis of orthonormal eigenvectors). It is Markov because all entries are non-negative and the columns add to 1.

\( A \) does not have an \( LU \)-decomposition, because we must do a row swap in reducing \( A \). It does have a \( QR \)-decomposition because the columns are linearly independent. It is diagonalizable, so it has an \( SAS^{-1} \) decomposition. Because it is also symmetric, the diagonalization actually gives a \( Q\Lambda Q^T \) decomposition.
$B$ is not invertible (it has rank 1). It is not orthogonal as it has no inverse. It is a projection matrix, because $B^2 = B$ and $B^T = B$ - in fact it projects onto the vector $(1,1,1)$. It is not a permutation matrix. It is diagonalizable because it is symmetric. It is Markov.

$B$ does have an $LU$-decomposition, since we do not need a row swap. It doesn’t have a $QR$-decomposition because the columns are dependent. It is diagonalizable and symmetric, so it has both a $SAS^{-1}$ and a $QAQ^T$ factorization.

**Problem 7:** Do problems 3 and 4 from section 10.2 (pg. 492).

**Solution** (10 points)

We solve the equation $Az = 0$ by reducing:

$$
\begin{bmatrix}
i & 1 & i \\
1 & i & i
\end{bmatrix} \sim
\begin{bmatrix}
i & 1 & i \\
0 & 2i & i-1
\end{bmatrix} \sim
\begin{bmatrix}
1 & -i & 1 \\
0 & 1 & (i-1)/2i
\end{bmatrix} \sim
\begin{bmatrix}
1 & 0 & (i+1)/2 \\
0 & 1 & (i+1)/2
\end{bmatrix}
$$

This matrix has one free column, so we get one special solution

$$(-i+1)/2, -(i+1)/2, 1)$$

I’ll rescale to use $z = (i + 1, i + 1, -2)$ instead, it doesn’t make any difference. The matrix $A^H$ is

$$A^H = A^T = \begin{bmatrix}
-i & 1 \\
1 & -i \\
-i & -i
\end{bmatrix}$$

Column 1 is $[-i, 1, -i]^T$. To calculate $C_1 \cdot z$, we need to take

$$C_1 \cdot z = C_1^H z = [i, 1, i] \begin{bmatrix}
i + 1 \\
i + 1 \\
-2
\end{bmatrix} = 0$$

Similarly,

$$C_2 \cdot z = C_2^H z = [1, i, i] \begin{bmatrix}
i + 1 \\
i + 1 \\
-2
\end{bmatrix} = 0$$
Of course these equations must be true; by taking the Hermitian of a column of $A^H$, we are just getting a row of $A$, and we know that any row of $A$ times a vector in the nullspace gives 0.

The matrix $A^T$ is

$$A^T = \begin{bmatrix} i & 1 \\ 1 & i \\ i & i \end{bmatrix}$$

These columns are not perpendicular to $z$, for example

$$C_1 \cdot z = C_1^H z = [-i, 1, -i] \begin{bmatrix} i + 1 \\ i + 1 \\ -2 \end{bmatrix} = 2 + 2i$$

Putting all this together, we see that the four fundamental spaces should be $C(A)$, $N(A)$, $C(A^H)$ and $N(A^H)$. They will satisfy the same orthogonal relationships as before: $N(A)$ and $C(A^H)$ are orthogonal complements, and $C(A)$ and $N(A^H)$ are orthogonal complements.

**Problem 8:** Do problem 15 from section 10.2 (pg. 493).

**Solution** (10 points)

Since $A$ is Hermitian, we expect it to have real eigenvalues, and a unitary eigenvector matrix $U$.

$A$ has eigenvalues given by the equation $\lambda^2 - \lambda - 2 = 0$, so we find eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$. The corresponding normalized eigenvectors are $x_1 = \frac{1}{\sqrt{6}}[1 - i, 2]^T$ and $x_2 = \frac{1}{\sqrt{3}}[i - 1, 1]^T$. (Another choice in the same direction is $\frac{1}{\sqrt{6}}[-2, 1 + i]^T$, which is more symmetric-looking.) So we obtain

$$\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$U = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 - i & -2 \\ 2 & 1 + i \end{bmatrix}$$

Since the columns of $U$ are (complex) orthogonal unit vectors, $U$ is unitary.

**Problem 9:** Do problem 6 in section 10.3 (pg. 500).

**Solution** (10 points)
The Fourier matrix $F_4$ is

$$F_4 = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & i & -1 & -i \\
1 & -1 & 1 & -1 \\
1 & -i & -1 & i
\end{bmatrix}$$

We can multiply this out to find

$$F_4^2 = \begin{bmatrix}
4 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 \\
0 & 4 & 0 & 0
\end{bmatrix}$$

and

$$F_4^4 = \begin{bmatrix}
16 & 0 & 0 & 0 \\
0 & 16 & 0 & 0 \\
0 & 0 & 16 & 0 \\
0 & 0 & 0 & 16
\end{bmatrix}$$

**Problem 10:** Do problem 11 in section 10.3 (pg. 501).

**Solution** (10 points)

Multiplying the two given matrices, we find that the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = i$, $\lambda_3 = i^2 = -1$, and $\lambda_4 = i^3 = -i$. 