18.06 Professor Strang Quiz 3 May 2, 2008

Your PRINTED name is: ____________________

Grading

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2

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Please circle your recitation:

1)	M 2	2-131	A. Ritter	2-085	2-1192	afr
2)	M 2	4-149	A. Tievsky	2-492	3-4093	tievsky
3)	M 3	2-131	A. Ritter	2-085	2-1192	afr
4)	M 3	2-132	A. Tievsky	2-492	3-4093	tievsky
5)	T 11	2-132	J. Yin	2-333	3-7826	jbyin
6)	T 11	8-205	A. Pires	2-251	3-7566	arita
7)	T 12	2-132	J. Yin	2-333	3-7826	jbyin
8)	T 12	8-205	A. Pires	2-251	3-7566	arita
9)	T 12	26-142	P. Buchak	2-093	3-1198	pmb
10)	T 1	2-132	B. Lehmann	2-089	3-1195	lehmann
11)	T 1	26-142	P. Buchak	2-093	3-1198	pmb
12)	T 1	26-168	P. McNamara	2-314	4-1459	petermc
13)	T 2	2-132	B. Lehmann	2-089	2-1195	lehmann
14)	T 2	26-168	P. McNamara	2-314	4-1459	petermc

1 (40 pts.) The (real) matrix A is

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & x & 3 \\ 2 & 3 & 6 \end{bmatrix}.$$

(a) What can you tell me about the eigenvectors of A?

What is the sum of its eigenvalues?

(b) For which values of x is this matrix A positive definite?

(c) For which values of x is A^2 positive definite? **Why**?

(d) If R is any **rectangular** matrix, *prove* from $x^{\mathrm{T}}(R^{\mathrm{T}}R)x$ that $R^{\mathrm{T}}R$ is positive semidefinite (or definite). What condition on R is the test for $R^{\mathrm{T}}R$ to be positive definite?

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2 (30 pts.) The **cosine** of a matrix is defined by copying the series for $\cos x$ (which always converges):

$$\cos A = I - \frac{1}{2!}A^2 + \frac{1}{4!}A^4 - \cdots$$

(a) Suppose $Ax = \lambda x$. Show that x is an eigenvector of $\cos A$. Find the eigenvalue.

(b) Find the eigenvalues of $A = \frac{\pi}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. The eigenvectors are $(1, 1)$ and $(1, -1)$. From the eigenvalues and eigenvectors of $\cos A$, find that matrix $\cos A$.

(c) The second derivative of the series for $\cos(At)$ is $-A^2 \cos(At)$. So $u(t) = \boldsymbol{\cos(At)u(0)}$ is a short formula for the solution of

$$\frac{d^2u}{dt^2} = -A^2 u \text{ starting from } u(0) \text{ with } u'(0) = 0\,.$$

Now construct that $u(t) = \cos(At)u(0)$ by the usual three steps when A is diagonalizable: $Ax_1 = \lambda_1 x_1$, $Ax_2 = \lambda_2 x_2$, $Ax_3 = \lambda_3 x_3$.

1. Expand $u(0) = c_1 x_1 + c_2 x_2 + c_3 x_3$ in the eigenvectors.
2. Multiply those eigenvectors by ____ , ____ , ____ .
3. Add up the solution $u(t) = c_1$ ____ $x_1 + c_2$ ____ $x_2 + c_3$ ____ x_3.

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3 (30 pts.) Suppose the vectors x, y give an orthonormal basis for $\mathbf{R}^2$ and $A = xy^{\mathrm{T}}$.

(a) Compute the rank of A and the rank of $A^2 = (xy^{\mathrm{T}})(xy^{\mathrm{T}})$. Use this information to find the eigenvalues of A.

(b) Explain why this matrix B is **similar** to A (*and write down what similar means*):

$$B = \begin{bmatrix} x^{\mathrm{T}} \\ y^{\mathrm{T}} \end{bmatrix} A \begin{bmatrix} x & y \end{bmatrix}$$

(c) The eigenvalues of Q are $\lambda_1 = e^{i\theta} = \cos\theta + i\sin\theta$ and $\lambda_2 = e^{-i\theta} = \cos\theta - i\sin\theta$:

$$\text{Rotation matrix } \; Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Find the eigenvectors of Q. Are they perpendicular?

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