Problem 1:
A sequence of numbers \( f_0, f_1, f_2, \ldots \) is defined by the recurrence

\[
f_{k+2} = 3f_{k+1} - f_k,
\]
with starting values \( f_0 = 1, f_1 = 1 \). (Thus, the first few terms in the sequence are
\(1, 1, 2, 5, 13, 34, 89, \ldots\))

(a) Defining \( u_k = \begin{pmatrix} f_{k+1} \\ f_k \end{pmatrix} \), re-express the above recurrence as \( u_{k+1} = A u_k \), and give the matrix \( A \).

(b) Find the eigenvalues of \( A \), and use these to predict what the ratio \( f_{k+1}/f_k \) of successive terms in the sequence will approach for large \( k \).

(c) The sequence above starts with \( f_0 = f_1 = 1 \), and \( |f_k| \) grows rapidly with \( k \). Keep \( f_0 = 1 \), but give a different value of \( f_1 \) that will make the sequence (with the same recurrence \( f_{k+2} = 3f_{k+1} - f_k \)) approach zero \( (f_k \to 0) \) for large \( k \).

Solution (18 points = 6+6+6)
(a) We have

\[
\begin{pmatrix} f_{k+2} \\ f_{k+1} \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{k+1} \\ f_k \end{pmatrix} \Rightarrow A = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}.
\]

(b) Eigenvalues of \( A \) are roots of \( \det(A - \lambda I) = \lambda^2 - 3\lambda + 1 = 0 \). They are \( \lambda_1 = \frac{3 + \sqrt{5}}{2} \) and \( \lambda_2 = \frac{3 - \sqrt{5}}{2} \). Note that \( \lambda_1 > \lambda_2 \), so the ratio \( f_{k+1}/f_k \) will approach \( \lambda_1 = \frac{3 + \sqrt{5}}{2} \) for large \( k \).

(c) Let \( v_1, v_2 \) be the eigenvectors with eigenvalues \( \lambda_1 \) and \( \lambda_2 \), respectively. So, we can write \( u_0 = c_1 v_1 + c_2 v_2 \) and then \( u_k = c_1 \lambda_1^k v_1 + c_2 \lambda_2^k v_2 \). If we need \( f_k \to 0 \),
we have to make \( c_1 = 0 \). In other words, \( u_0 \) must be proportional to the eigenvector \( v_2 \).

\[
A - \lambda_2 I = \begin{pmatrix}
\frac{3 + \sqrt{5}}{2} & -1 \\
1 & -\frac{3 - \sqrt{5}}{2}
\end{pmatrix} \implies v_2 = \begin{pmatrix}
\frac{3 - \sqrt{5}}{2} \\
1
\end{pmatrix}.
\]

Hence, we need to take \( f_1 = \frac{3 - \sqrt{5}}{2} \) so that \( f_k \) will approach zero for large \( k \).
Problem 2: For the matrix \( A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \) with rank 2, consider the system of equations \( Ax = b \).

(i) \( Ax = b \) has a solution whenever \( b \) is orthogonal to some nonzero vector \( c \). Explicitly compute such a vector \( c \). Your answer can be multiplied by any overall constant, because \( c \) is any basis for the space of \( A \).

(ii) Find the orthogonal projection \( p \) of the vector \( b = \begin{pmatrix} 9 \\ 9 \\ 9 \end{pmatrix} \) onto \( C(A) \). (Note: The matrix \( A^T A \) is singular, so you cannot use your formula \( P = A(A^T A)^{-1} A^T \) to obtain the projection matrix \( P \) onto the column space of \( A \). But I have repeatedly discouraged you from computing \( P \) explicitly, so you don’t need to be reminded anyway, right?)

(iii) If \( p \) is your answer from (ii), then a solution \( y \) of \( Ay = p \) minimizes what? [You need not answer (ii) or compute \( y \) for this part.]

Solution (18 points = 7+7+4)

(i) The system of equations \( Ax = b \) has a solution if and only if \( b \) lies in the column space of \( A \), which is orthogonal to the left nullspace of \( A \). We solve for a (nonzero) vector \( c \) in the left nullspace using Gaussian elimination, as follows.

\[
A^T = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \ c = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.
\]

The answer can by any nonzero multiple of \( c \), which will be a basis for the left nullspace of \( A \).

(ii) Method 1: Since \( c \) is a basis of the orthogonal complement of the column space \( C(A) \), the projection of \( b \) onto \( C(A) \) can be computed as

\[
p = b - \frac{c^T b c}{\|c\|^2} = \begin{pmatrix} 9 \\ 9 \\ 9 \end{pmatrix} - \frac{9}{3} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 12 \end{pmatrix}.
\]
Method 2: (not recommended) We know that $p$ is the best linear approximation of $b$. So we solve

$$A^TA \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = A^T \begin{pmatrix} 9 \\ 9 \end{pmatrix},$$

$$\begin{pmatrix} 6 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 36 \\ 18 \\ 0 \end{pmatrix}.$$ 

We can get a particular solution $y = (6, 0, 0)^T$. (There are other solutions too.) Hence,

$$p = A \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 12 \end{pmatrix}.$$

(iii) Since $p$ is the orthogonal projection of $b$ onto $C(A)$, a solution $y$ of $Ay = p$ minimizes the distance $\|Ay - b\|$.
Problem 3: True or false. Give a counter-example if false. (You need not provide a reason if true.)

(a) If $Q$ is an orthogonal matrix, then $\det Q = 1$.

(b) If $A$ is a Markov matrix, then $du/dt = Au$ approaches some finite constant vector (a “steady state”) for any initial condition $u(0)$.

(c) If $S$ and $T$ are subspaces of $\mathbb{R}^2$, then their intersection (points in both $S$ and $T$) is also a subspace.

(d) If $S$ and $T$ are subspaces of $\mathbb{R}^2$, then their union (points in either $S$ or $T$) is also a subspace.

(e) The rank of $AB$ is less than or equal to the ranks of $A$ and $B$ for any $A$ and $B$.

(f) The rank of $A + B$ is less than or equal to the ranks of $A$ and $B$ for any $A$ and $B$.

Solution (12 points = 2+2+2+2+2+2)

(a) False. For example, $Q = (-1)$ is an orthogonal matrix: $Q^TQ = (-1)(-1) = (1)$.

REMARK: In general, for a real orthogonal matrix $Q$, $\det Q = \pm 1$. This is because $\det(Q^TQ) = \det(I) = 1 \Rightarrow \det(Q)^2 = \det(Q^T)\det(Q) = 1$.

(b) False. Be careful here that we are discussing differential equations but not the powers of $A$. For example, $A = (1)$, the differential equation has solution $u = ce^t$ for some constant $c$, which does not approach to any finite constant vector.

REMARK: It is true that for the Markov process $u_{k+1} = Au_k$, $u_k$ approaches some finite constant vector (a “steady state”) for any initial condition $u_0$.

(c) True. Intersections of subspaces are always subspaces.

(d) False. For example, $S$ and $T$ are the $x$- and $y$-axes. Then $(1, 1) = (1, 0) + (0, 1)$ is a linear combination of points in the union of $S$ and $T$, but does not lie in the union itself. So the union of $S$ and $T$ is not a subspace.

(e) True. One may see this by arguing as follows. Since the column space of $AB$ is a subspace of the column space of $A$, the rank of $AB$ is less than or equal to
the rank of $A$. Similarly, since the row space of $AB$ is a subspace of the row space of $B$, the rank of $AB$ is less than or equal to the rank of $B$.

(f) False. $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ both have rank 1. But $A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has rank 2.

REMARK: It is true that $\text{rank}(A + B) \leq \text{rank}A + \text{rank}B$. 
**Problem 4:** Consider the matrix

\[
A = \begin{pmatrix}
1 & 1 & 1 \\
1 & -1 & -1 \\
1 & 0 & -3 \\
1 & 0 & -1 \\
\end{pmatrix}
\]

(a) Find an orthonormal basis for \( C(A) \) using Gram-Schmidt, forming the columns of a matrix \( Q \).

(b) Write each step of your Gram-Schmidt process from (a) as a multiplication of \( A \) on the \( \text{ left or right} \) by some invertible matrix. Explain how the product of these invertible matrices relates to the matrix \( R \) from the QR factorization \( A = QR \) of \( A \).

(c) Gram-Schmidt on another matrix \( B \) (of the same size as \( A \)) gives the same orthonormal basis (the same \( Q \)) as in part (a). Which of the four subspaces, if any, must be the same for the matrices \( AA^T \) and \( BB^T \)? [\textit{You can do this part without doing (a) or (b).}]

**Solution** (18 points = 6+6+6)

(a) From \( \mathbf{u}_1 = (1, 1, 1, 1)^T \), we get \( \mathbf{q}_1 = \mathbf{u}_1 / \| \mathbf{u}_1 \| = \frac{1}{2}(1, 1, 1, 1)^T \).

\[
\begin{align*}
\mathbf{v}_2 &= (1, -1, 0, 0)^T, \\
\mathbf{u}_2 &= \mathbf{v}_2 - \mathbf{q}_1^T \mathbf{v}_2 \mathbf{q}_1 = \mathbf{v}_2 = (1, -1, 0, 0)^T, \\
\mathbf{q}_2 &= \mathbf{v}_2 / \| \mathbf{v}_2 \| = \frac{1}{\sqrt{2}}(1, -1, 0, 0)^T; \\
\mathbf{v}_3 &= (1, -1, -3, -1)^T, \\
\mathbf{u}_3 &= \mathbf{v}_3 - \mathbf{q}_1^T \mathbf{v}_3 \mathbf{q}_1 - \mathbf{q}_2^T \mathbf{v}_3 \mathbf{q}_2 = \mathbf{v}_3 + \mathbf{u}_1 - \mathbf{u}_2 = (1, 1, -2, 0)^T, \\
\mathbf{q}_3 &= \mathbf{v}_3 / \| \mathbf{v}_3 \| = \frac{1}{\sqrt{6}}(1, 1, -2, 0)^T.
\end{align*}
\]

Hence, we have

\[
Q = \begin{pmatrix}
1/2 & 1/\sqrt{2} & 1/\sqrt{6} \\
1/2 & -1/\sqrt{2} & 1/\sqrt{6} \\
1/2 & 0 & -2/\sqrt{6} \\
1/2 & 0 & 0 \\
\end{pmatrix}
\]
(b) Each step of the Gram Schmidt process from (a) is a multiplication of $A$ on the right as follows.

\[
A \sim A \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim A \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\sim A \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \\
\sim A \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\sqrt{6} \end{pmatrix} = Q.
\]

The product of these invertible $3 \times 3$ matrices is exactly $R^{-1}$.

(c) Since the Gram-Schmidt of $A$ and $B$ gives the same outcome, the column space of $A$ and $B$ are the same. We know that $A$ and $AA^T$ have the same column space, and $B$ and $BB^T$ have the same column space. Hence $AA^T$ and $BB^T$ have the same column space. Moreover, since left nullspace is always orthogonal to the column space, $AA^T$ and $BB^T$ have the same left nullspace too. Also, notice that $AA^T$ and $BB^T$ are symmetric matrices, their row spaces are the same as the column spaces, and their nullspaces are the same as the left nullspaces. Therefore, all four subspaces of $AA^T$ are the same as $BB^T$. 
Problem 5: The complete solution to $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

for any arbitrary constants $c$ and $d$.

(i) If $A$ is an $m \times n$ matrix with rank $r$, give as much true information as possible about the integers $m$, $n$, and $r$.

(ii) Construct an explicit example of a possible matrix $A$ and a possible right-hand side $\mathbf{b}$ with the solution $\mathbf{x}$ above. (There are many acceptable answers; you only have to provide one.)

Solution (16 points = 8+8)

(i) Since we can multiply $A$ with $\mathbf{x}$, $n = 3$. Also, since the nullspace of $A$ is 2-dimensional, $r = n - 2 = 1$. There is no restriction on $m$ except that $m \geq r = 1$.

(ii) We construct a minimal one, namely, $A = (a_1 \ a_2 \ a_3)$ is $1 \times 3$. For this, we need $A \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0$ and $A \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = 0$. That is

$$\begin{pmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$ 

A special solution is $A = (1 \ -1 \ 2)$. In this case, $\mathbf{b} = A\mathbf{x} = (1 \ -1 \ 2) \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = (-1)$. So, an example is

$$(1 \ -1 \ 2) \mathbf{x} = (-1).$$
Problem 6: Consider the matrix

\[ A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \]

(i) \( A \) has one eigenvalue \( \lambda = -1 \), and the other eigenvalue is a double root of \( \det(A - \lambda I) \). What is the other eigenvalue? (Very little calculation required.)

(ii) Is \( A \) defective? Why or why not?

(iii) Using the above \( A \), suppose we want to solve the equation

\[ \frac{du}{dt} = Au + cu \]

where \( c \) is some real number, for some initial condition \( u(0) \).

(a) For what values of \( c \) will the solutions \( u(t) \) always go zero as \( t \to \infty \)?

(b) For what values of \( c \) will the solutions \( u(t) \) typically diverge (\( ||u(t)|| \to \infty \)) as \( t \to \infty \)?

(c) For what values of \( c \) will the solutions \( u(t) \) approach a constant vector (possibly zero) as \( t \to \infty \)?

Solution

(18 points = 6+6+6 (2+2+2))

(i) Let \( \lambda_1 = -1 \) and let \( \lambda_2 = \lambda_3 \) denote the double roots. Then from the trace of \( A \), we have \( \lambda_1 + 2\lambda_2 = \text{trace}(A) = 3 \). Hence, \( \lambda_2 = 2 \).

(ii) \( A \) is not defective. There are two ways to see it. For one way, since \( A \) is symmetric, it is always non-defective; for another way, we compute \( A - \lambda_2 I = \begin{pmatrix} -1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \), which has rank 1 and hence its nullspace is 2-dimentional.

(iii) The key point here is that \( A + cI \) would have eigenvalues \( \lambda_1 + c \) and \( \lambda_2 + c \) (with multiplicity 2). An alternative point of view is as follows. If we write the initial condition \( u(t) = c_1(t)v_1 + c_2(t)v_2 + c_3(t)v_3 \), then the differential equation becomes

\[ \frac{dc_1(t)}{dt}v_1 + \frac{dc_2(t)}{dt}v_2 + \frac{dc_3(t)}{dt}v_3 = c_1\lambda_1 v_1 + c_2\lambda_2 v_2 + c_3\lambda_3 v_3 + cc_1 v_1 + cc_2 v_2 + cc_3 v_3. \]
We have

\[
\begin{align*}
\frac{dc_1(t)}{dt} &= c_1 \lambda_1 + cc_1, \quad \Rightarrow \quad c_1 = e^{(\lambda_1 + c)t}; \\
\frac{dc_2(t)}{dt} &= c_2 \lambda_2 + cc_2, \quad \Rightarrow \quad c_2 = e^{(\lambda_2 + c)t}; \\
\frac{dc_3(t)}{dt} &= c_3 \lambda_3 + cc_3, \quad \Rightarrow \quad c_3 = e^{(\lambda_3 + c)t};
\end{align*}
\]

(a) If we require \( u(t) \) always go zero as \( t \to \infty \), \( \lambda_1 + c < 0, \lambda_2 + c = \lambda_3 + c < 0 \). Hence, we require \( c < -2 \).

(b) If the solution \( u(t) \) typically diverge, we need either \( \lambda_1 + c > 0 \) or \( \lambda_2 + c = \lambda_3 + c > 0 \). Hence, we require \( c > -2 \).

(c) If we allow the solution to approach to some constant vector, we allow to have the extreme case of (a), that is to say \( c \leq -2 \).