18.06 Problem Set 4

Due Wednesday, 11 March 2008 at 4pm in 2-106.

1. A is an $m \times n$ matrix of rank $r$. Suppose there are right-hand-sides $b$ for which $Ax = b$ has no solution.
   (a) What are all the inequalities ($<$ or $\leq$) that must be true between $m$, $n$, and $r$?
   (b) $A^T y = 0$ has solutions other than $y = 0$. Why must this be true?

2. $A$ is an $m \times n$ matrix of rank $r$. Which of the four fundamental subspaces are the same for:
   (a) $A$ and $(A A)$
   (b) $(A A)$ and $(A A A A)$
   Explain why all three matrices $A$, $(A A)$, and $(A A A A)$ must have the same rank $r$.

3. Find a basis for each of the four subspaces for
   $$ A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} $$

4. True or false (give a reason if true, or a counterexample if false):
   (a) $A$ and $A^T$ have the same number of pivots
   (b) $A$ and $A^T$ have the same left nullspace
   (c) If the $C(A) = C(A^T)$, then $A = A^T$.
   (d) If $A^T = -A$, then the row space of $A$ is the same as the column space of $A$.

5. Use the Matlab command $A = \text{rand}(10,5)$; to make a random $10 \times 5$ matrix $A$, and $B = \text{rand}(5,9)$ to make a random $5 \times 9$ matrix $B$. Then use the command $[R, p] = \text{rref}(A*B)$; to find the row-reduced echelon form $R$ and a list $p$ of the pivot columns. Using this information, give bases for the nullspace, column space, and row space of $AB$.

6. Explain why the following statement must be true: if a subspace $S$ is contained in another subspace $V$, then the orthogonal complement $V^\perp$ is contained in the orthogonal complement $S^\perp$.

7. If $A^T A x = 0$ then $A x = 0$. Reason: $A^T A x = 0$ means that $A x$ in the nullspace of ________. $A x$ is also in the ________ space of $A$. These two spaces are ________, so their only intersection is $A x = 0$. Thus, $A^T A$ has the same nullspace as $A$. (We derive this in another way in class.)

8. Suppose you have two matrices $V$ and $W$ such that $C(V)$ and $C(W)$ are orthogonal subspaces. What is $V^T W$?

9. Suppose $L$ is a one-dimensional subspace (a line through the origin) in $\mathbb{R}^3$. Its orthogonal complement $L^\perp$ is the ________ perpendicular to $L$. Then $(L^\perp)^\perp$ is a ________ perpendicular to $L^\perp$, and in fact $(L^\perp)^\perp$ is the same as ________.

10. Let $N$ be a matrix whose columns are a basis for the nullspace of $A$. Then the nullspace of $N^T$ is the ________ space of $A$. 

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11. Let $A$ be the matrix 

$$A = \begin{pmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{pmatrix}.$$ 

(a) Find the projection matrix $P_C$ onto $C(A)$.
(b) Find the projection matrix $P_R$ onto the row space of $A$.
(c) Compute $P_C A P_R$. Explain your result.
(d) For any matrix $A$ (not necessarily the one above), with $P_C$ and $P_R$ defined as the projection matrices onto $A$’s column and row space respectively, conclude that you would get $P_C A P_R =$

12. Find the projection matrix $P$ onto the plane $x + 2y - z = 0$ in two ways:

(a) Choose two vectors in the plane and make them the columns of a matrix $A$. The plane is the column space. Then compute $P = A(A^T A)^{-1} A^T$.
(b) Write a vector $e$ that is perpendicular to that plane. Compute the matrix $Q = ee^T / e^T e$ that projects onto the $e$ direction. Then compute $P = I - Q$.

13. The nullspace of $A^T$ is ____________ to the column space $C(A)$, so if $A^T b = 0$ then the projection of $b$ onto $C(A)$ should be $p =$ ____________. Check that $Pb$ gives this answer, where $P$ is the projection matrix $P = A(A^T A)^{-1} A^T$.

14. Explain why one must have $P^2 = P$, from the definition of the projection matrix $P$ onto the column space of a matrix $A$ (if we take a vector $b$ and project it to the column space to get $Pb$, then project it again, we must get ____________). Check explicitly that this is true from the formula $P = A(A^T A)^{-1} A^T$. 

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