18.06 Quiz 2 Professor Strang April 6, 2011

Your PRINTED name is ____________________ 1.

Your Recitation Instructor (and time) is ____________________ 2.

Instructors: (Pires)(Hezari)(Sheridan)(Yoo) 3.

Please show enough work so we can see your method and give due credit.

1. **(8 pts. each)** Suppose a_1 and a_2 are orthogonal unit vectors in R^5.

   (a) What are the requirements on a matrix P to be a projection matrix? Verify that $P = a_1 a_1^T + a_2 a_2^T$ satisfies those requirements.

   (b) If a_3 is in R^5, what combination of a_1 and a_2 is closest to a_3?

   (c) Find a combination c of a_1, a_2, a_3 that is perpendicular to a_1 and a_2. *If possible, choose* $c \neq 0$. Describe all cases when $c = 0$ is the only possibility.

   (d) Show that a_1 and a_2 and c are eigenvectors of P (*if* $c \neq 0$) and find their eigenvalues.

1:

a: P is a projection (orthogonal projection!) if

$$\begin{cases} P^2 = P, \\ P^T = P. \end{cases}$$

We know check this for $P = a_1 a_1^T + a_2 a_2^T$.

- $P^2 = (a_1 a_1^T + a_2 a_2^T)(a_1 a_1^T + a_2 a_2^T) = a_1 \underbrace{a_1^T a_1} a_1^T + a_1 \underbrace{a_1^T a_2} a_2^T + a_2 \underbrace{a_2^T a_1} a_1^T + a_2 \underbrace{a_2^T a_2} a_2^T$.

  Since $a_1^T a_2 = 0$, $a_2^T a_1 = 0$, and $a_1^T a_1 = a_2^T a_2 = 1$, we get

  $P^2 = a_1 a_1^T + a_2 a_2^T = P$.

- $P^T = (a_1 a_1^T + a_2 a_2^T)^T = (a_1^T)^T a_1^T + (a_2^T)^T a_2^T = a_1 a_1^T + a_2 a_2^T = P$.

b: The closest combination is $Pa_3 = (a_1^T a_3) a_1 + (a_2^T a_3) a_2$.

c: $c =$ error term $= a_3 - Pa_3 = a_3 - (a_1^T a_3) a_1 - (a_2^T a_3) a_2$.

$c = 0$ only if c is in the plane generated by a_1 and a_2.

d: Since P is the projection on the column space of

$A = [a_1 | a_2]$, we have:

$Pa_1 = a_1 \Longrightarrow \lambda_1 = 1$

$Pa_2 = a_2 \Longrightarrow \lambda_2 = 1$

$Pc = 0 \Longrightarrow \lambda_3 = 0$.

2. **(7 pts. each)**

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 9 & 10 \\ 0 & 0 & 11 & 12 \end{bmatrix}.$$

(a) Find all nonzero terms in the big formula $\det A = \sum \pm a_{1\alpha}\, a_{2\beta}\, a_{3\gamma}\, a_{4\delta}$ and combine them to compute $\det A$.

(b) Find all the pivots of A.

(c) Find the cofactors C_{11}, C_{12}, C_{13}, C_{14} of row 1 of A.

(d) Find column 1 of A^{-1}.

2:

a: $\det A = 1\,(6\cdot(9\cdot 12 - 10\cdot 11)) - 2\,(5\,(9\cdot 12 - 10\cdot 11)) = 8$

b: By row reduction:

A reduces to $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -16 \\ 0 & 0 & 9 & 10 \\ 0 & 0 & 0 & -\frac{12}{9} \end{bmatrix}$.

Hence the pivots are: $1, -4, 9, -\frac{12}{9}$.

c:

$$C_{11} = \det \begin{bmatrix} 6 & 7 & 8 \\ 0 & 9 & 10 \\ 0 & 11 & 12 \end{bmatrix} = -12$$

$$C_{12} = -\det \begin{bmatrix} 5 & 7 & 8 \\ 0 & 9 & 10 \\ 0 & 11 & 12 \end{bmatrix} = 10$$

$$C_{13} = \det \begin{bmatrix} 5 & 6 & 8 \\ 0 & 0 & 10 \\ 0 & 0 & 12 \end{bmatrix} = 0$$

$$C_{14} = -\det \begin{bmatrix} 5 & 6 & 7 \\ 0 & 0 & 9 \\ 0 & 0 & 11 \end{bmatrix} = 0.$$

d: From c we get

$$(A^{-1})_{11} = \frac{-12}{8}$$

$$(A^{-1})_{21} = \frac{10}{8}$$

$$(A^{-1})_{31} = 0$$

$$(A^{-1})_{41} = 0.$$

3. **(8 pts. each)** Suppose A is a 2 by 2 matrix and $Ax = x$ and $Ay = -y$ ($x \neq 0$ and $y \neq 0$).

(a) (Reverse engineering) What is the polynomial $p(\lambda) = \det(A - \lambda I)$?

(b) If you know that the first column of A is $(2, 1)$, find the second column:

$$A = \begin{bmatrix} 2 & ? \\ 1 & ? \end{bmatrix}.$$

(c) For that matrix in part (b), find an invertible S and a diagonal matrix Λ so that $A = S\Lambda S^{-1}$.

(d) Compute A^{101}. (If you don't solve parts (b) -(c), use the description of A at the start. In all questions **show enough work** so we can see your method and give due credit.)

(e) If $Ax = x$ and $Ay = -y$ (with $x \neq 0$ and $y \neq 0$) prove that x and y are *independent.*

Start of a proof: Suppose $z = cx + dy = 0$. Then $Az =$ (follow from here.)

3:

a: $p(\lambda) = (1-\lambda)(-1-\lambda) = \lambda^2 - 1$

b: we know that $Tr A = 1 + (-1) = 0$.
on the other hand if we put $A = \begin{bmatrix} 2 & a_{12} \\ 1 & a_{22} \end{bmatrix}$
then $Tr A = 2 + a_{22}$. Hence $a_{22} = -2$.
To find a_{12} we note that on one hand
$\det A = 1 \cdot (-1) = -1$ and on the other hand
$\det A = 2a_{22} - a_{12} = -4 - a_{12}$. Therefore $a_{12} = -3$.
So $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$.

(c): It is easy to see that X an eigenvector of $\lambda_1 = 1$
is $X = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and for Y an eigevector of $\lambda_2 = -1$
we have $Y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So we can choose $S = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$.

d: From c we have $A^{101} = S\Lambda^{101}S^{-1} = S\Lambda S^{-1} = A$.
Note that $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and therefore $\Lambda^{101} = \Lambda$.

e: on one hand since $Z = 0$ we have $AZ = 0$.
on the other hand $AZ = A(cX + dY) = cAX + dAY$
$= cX - dY$.

Therefore

$\begin{cases} Z = cX + dY = 0 \\ AZ = cX - dY = 0 \end{cases} \xrightarrow{\text{since } X \neq 0,\ Y \neq 0} c = d = 0 \Rightarrow$ X and Y are linearly independent.