18.06 Solutions to PSet 4

3.5:

16: These bases are not unique! (a) $(1, 1, 1, 1)$ for the space of all constant vectors $(c, c, c, c)$ (b) $(1, -1, 0, 0), (1, 0, -1, 0), (0, 0, 0, -1)$ for the space of vectors with sum of components $= 0$ (c) $(1, -1, -1, 0), (1, -1, 0, -1)$ for the space perpendicular to $(1, 1, 0, 0)$ and $(1, 0, 1, 1)$ (d) The columns of $I$ are a basis for its column space, the empty set is a basis (by convention) for $N(I) = \{\text{zero vector}\}$.

26:

(a) \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad 
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad 
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

(b) Add \[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad 
\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}, \quad 
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad 
\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{bmatrix}, \quad 
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & -1 & 0
\end{bmatrix}
\]

These are simple bases (among many others) for (a) diagonal matrices (b) symmetric matrices (c) skew-symmetric matrices. The dimensions are 3, 6, 3.

30: \[
\begin{bmatrix}
-1 & 2 & 0 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{bmatrix}, \quad 
\begin{bmatrix}
-1 & 0 & 2 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{bmatrix}, \quad 
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{bmatrix}
\]

41: \[I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \]
The six $P$'s are dependent.

Those five are independent: The 4th has $P_{11} = 1$ and cannot be a combination of the others. Then the 2nd cannot be (from $P_{32} = 1$) and also 5th ($P_{32} = 1$). Continuing, a nonzero combination of all five could not be zero. Further challenge: How many independent 4 by 4 permutation matrices?

3.6:

6: $A$: dim 2, 2, 1: Rows $(0, 3, 3, 3)$ and $(0, 1, 0, 1)$; columns $(3, 0, 1)$ and $(3, 0, 0)$; nullspace $(1, 0, 0, 0)$ and $(0, -1, 0, 1)$; $N(A^T) (0, 1, 0)$. $B$: dim 1, 1, 0, 2 Row space $(1)$, column space $(1, 4, 5)$, nullspace: empty basis, $N(A^T) (-4, 1, 0)$ and $(-5, 0, 1)$.

14: Row space basis can be the nonzero rows of $U$: $(1, 2, 3, 4), (0, 1, 2, 3), (0, 0, 1, 2)$; nullspace basis $(0, 1, -2, 1)$ as for $U$; column space basis $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ (happen to have $C(A) = C(U) = \mathbb{R}^3$); left nullspace has empty basis.

16: If $Av = 0$ and $v$ is a row of $A$ then $v \cdot v = 0$.

32: The key is equal row spaces. First row of $A$ = combination of the rows of $B$: only
possible combination (notice $I$) is 1 (row 1 of $B$). Same for each row so $F = G$.

8.2: 

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad \text{leads to} \quad x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{solving} \quad A^T y = 0.$$ 

9: Elimination on $Ax = b$ always leads to $y^T b = 0$ in the zero rows of $U$ and $R$: $-b_1 + b_2 - b_3 = 0$ and $b_3 - b_4 + b_5 = 0$ (those $y$’s are from Problem 8 in the left nullspace). This is Kirchhoff’s Voltage Law around the two loops.

12.a: The nullspace and rank of $A^T A$ and $A$ are always the same.

4.1: 

9: $Ax$ is always in the column space of $A$. If $A^T Ax = 0$ then $Ax$ is also in the nullspace of $A^T$. So $Ax$ is perpendicular to itself. Conclusion: $Ax = 0$ if $A^T Ax = 0$.

11: For $A$: The nullspace is spanned by $(-2, 1)$, the row space is spanned by $(1, 2)$. The column space is the line through $(1, 3)$ and $N(A^T)$ is the perpendicular line through $(3, -1)$. For $B$: The nullspace of $B$ is spanned by $(0, 1)$, the row space is spanned by $(1, 0)$. The column space and left nullspace are the same as for $A$.

22: $(1, 1, 1, 1)$ is a basis for $P^\perp$. $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ has $P$ as its nullspace and $P^\perp$ as row space.

33: Both $r$’s orthogonal to both $n$’s, both $c$’s orthogonal to both $\ell$’s, each pair independent. All $A$’s with these subspaces have the form $[c_1 \ c_2 M r_1 \ r_2]^T$ for a 2 by 2 invertible $M$.

4.2: 

16: $\frac{1}{2}(1, 2, -1) + \frac{3}{2}(1, 0, 1) = (2, 1, 1)$. So $b$ is in the plane. Projection shows $Pb = b$. 