18.06 Solutions to PSet 5

4.2:

11: (a) \( p = A(A^T A)^{-1} A^T b = (2, 3, 0), \) \( e = (0, 0, 4) \), \( A^T e = 0 \)  
(b) \( p = (4, 4, 6), \) \( e = 0 \).

12: \( P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \) is projection matrix onto the column space of \( A \) (the \( xy \) plane).

\( P_2 = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \) is projection matrix onto the second column space. Certainly \((P_2)^2 = P_2\).

17: If \( P^2 = P \) then \((I - P)^2 = (I - P)(I - P) = I - PI - IP + P^2 = I - P\). When \( P \) projects onto the column space, \( I - P \) projects onto the left nullspace.

26: \( A^{-1} \) exists since the rank is \( r = m \). Multiply \( A^2 = A \) by \( A^{-1} \) to get \( A = I \).

32: Since \( P_1 b \) is in \( C(A) \), \( P_2(P_1 b) \) equals \( P_1 b \). So \( P_2 P_1 = P_1 = a a^T / a^T a \) where \( a = (1, 2, 0) \).

4.3:

9: Project \( b \) 4D to 3D

12: (a) \( a = (1, \ldots, 1) \) has \( a^T a = m, \) \( a^T b = b_1 + \cdots + b_m \). Therefore \( \hat{x} = a^T b / m \) is the mean of the \( b \)'s. \( b = (1, 2, b) \) \( \|e\|^2 = \sum_{i=1}^m (b_i - \hat{x})^2 = \text{variance} \)

(c) \( p = (3, 3, 3), \) \( e = (-2, -1, 3) \) \( p^T e = 0 \), \( P = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \).

17: \( \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix} \). The solution \( \hat{x} = \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix} \) comes from \( \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix} \).

28: Only 1 plane contains \( 0, a_1, a_2 \) unless \( a_1, a_2 \) are dependent. Same test for \( a_1, \ldots, a_n \).

4.4:

4: (a) \( Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \), \( QQ^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \) \( \neq I\). Any \( Q \) with \( n < m \) has \( QQ^T \neq I \).

(b) \((1, 0)\) and \((0, 0)\) are orthogonal, not independent. Nonzero orthogonal vectors are independent. (c) Starting from \( q_1 = (1, 1, 1)/\sqrt{3} \) my favorite is \( q_2 = (1, -1, 0)/\sqrt{2} \) and \( q_3 = (1, 1, -2)/\sqrt{6} \).

12: (a) Orthonormal \( a \)'s: \( a^T b = a^T(x_1 a_1 + x_2 a_2 + x_3 a_3) = x_1(a^T a_1) = x_1 \)

(b) Orthonormal \( a \)'s: \( a^T b = a^T(x_1 a_1 + x_2 a_2 + x_3 a_3) = x_1(a^T a_1) \). Therefore \( x_1 = a^T b / a^T a_1 \)
(c) $x_1$ is the first component of $A^{-1}b$.

18: $A = a = (1, -1, 0, 0)$; $B = b - p = (\frac{1}{2}, \frac{1}{2}, -1, 0)$; $C = c - p_A - p_B = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1)$. Notice the pattern in those orthogonal $A, B, C$. In $R^3$, $D$ would be $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -1)$.

19: If $A = QR$ then $A^TA = R^TQ^TQR = R^TR$ = lower triangular times upper triangular (this Cholesky factorization of $A^TA$ uses the same $R$ as Gram-Schmidt!).

The example has $A = \begin{bmatrix} -1 & 1 \\ 2 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix} = QR$ and the same $R$ appears in $A^TA = \begin{bmatrix} 9 & 9 \\ 9 & 18 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = R^TR$.

24: (a) One basis for the subspace $S$ of solutions to $x_1 + x_2 + x_3 - x_4 = 0$ is $v_1 = (1, -1, 0, 0), v_2 = (1, 0, -1, 0), v_3 = (1, 0, 0, 1)$. (b) Since $S$ contains solutions to $(1, 1, 1, -1)^T x = 0$, a basis for $S^\perp$ is $(1, 1, 1, -1)$. (c) Split $(1, 1, 1, 1) = b_1 + b_2$ by projection on $S^\perp$ and $S$: $b_2 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $b_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2})$. 