18.06 Solutions to PSet 9

6.7:

3: If $A$ has rank 1 then so does $A^TA$. The only nonzero eigenvalue of $A^TA$ is its trace, which is the sum of all $a_{ij}^2$. (Each diagonal entry of $A^TA$ is the sum of $a_{ij}^2$ down one column, so the trace is the sum down all columns.) Then $\sigma_1$ is the square root of this sum, and $\sigma_1^2$ is the sum of all $a_{ij}^2$.

6: $AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has $\sigma_1^2 = 3$ with $u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $\sigma_2^2 = 1$ with $u_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$. $A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ has $\sigma_1^2 = 3$ with $v_1 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$, $\sigma_2^2 = 1$ with $v_2 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$, and $v_3 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$. Then $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} = [u_1 \quad u_2] \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} [v_1 \quad v_2 \quad v_3]^T$.

7: The matrix $A$ in Problem 6 had $\sigma_1 = \sqrt{3}$ and $\sigma_2 = 1$ in $\Sigma$. The smallest change to rank 1 is to make $\sigma_2 = 0$. In the factorization

$$A = U\Sigma V^T = u_1\sigma_1v_1^T + u_2\sigma_2v_2^T$$

this change $\sigma_2 \to 0$ will leave the closest rank–1 matrix as $u_1\sigma_1v_1^T$. See Problem 14 for the general case of this problem.

9: $A = UV^T$ since all $\sigma_j = 1$, which means that $\Sigma = I$.

10: A rank–1 matrix with $Av = 12u$ would have $u$ in its column space, so $A = uu^T$ for some vector $u$. I intended (but didn’t say) that $u$ is a multiple of the unit vector $v = \frac{1}{3}(1, 1, 1, 1)$ in the problem. Then $A = 12uv^T$ to get $Av = 12u$ when $v^Tu = 1$.

11: If $A$ has orthogonal columns $w_1, \ldots, w_n$ of lengths $\sigma_1, \ldots, \sigma_n$, then $A^TA$ will be diagonal with entries $\sigma_1^2, \ldots, \sigma_n^2$. So the $\sigma$’s are definitely the singular values of $A$ (as expected). The eigenvalues of that diagonal matrix $A^TA$ are the columns of $I$, so $V = I$ in the SVD. Then the $u_i$ are $Av_i/\sigma_i$ which is the unit vector $w_i/\sigma_i$.

The SVD of this $A$ with orthogonal columns is $A = U\Sigma V^T = (A\Sigma^{-1})(\Sigma)(I)$.

14: the smallest change in $A$ is to set its smallest singular value $\sigma_2$ to zero. See # 7.

15: The singular values of $A + I$ are not $\sigma_j + 1$. They come from eigenvalues of $(A + I)^T(A + I)$.

8.1:

3: The rows of the free-free matrix in equation (9) add to $[0 \quad 0 \quad 0]$ so the right side needs $f_1 + f_2 + f_3 = 0$. $f = (-1, 0, 1)$ gives $c_2u_1 - c_2u_2 = -1, c_3u_2 - c_3u_3 = -1, 0 = 0$. Then $u_{\text{particular}} = (-c_2^{-1} - c_3^{-1}, -c_3^{-1}, 0)$. Add any multiple of $u_{\text{nullspace}} = (1, 1, 1)$. 


4: $\int -\frac{d}{dx} \left( c(x) \frac{du}{dx} \right) \, dx = -\left[ c(x) \frac{du}{dx} \right]^1_0 = 0$ (bdry cond) so we need $\int f(x) \, dx = 0$.

7: For 5 springs and 4 masses, the 5 by 4 $A$ has two nonzero diagonals: all $a_{ii} = 1$ and $a_{i+1,i} = -1$. With $C = \text{diag}(c_1, c_2, c_3, c_4, c_5)$ we get $K = A^TCA$, symmetric tridiagonal with diagonal entries $K_{ii} = c_i + c_{i+1}$ and off-diagonals $K_{i+1,i} = -c_{i+1}$. With $C = I$ this $K$ is the $-1, 2, -1$ matrix and $K(2, 3, 3, 2) = (1, 1, 1, 1)$ solves $Ku = \text{ones}(4, 1)$. ($K^{-1}$ will solve $Ku = \text{ones}(4)$.)