1. Without asking anyone for help, write down an accurate definition of what it means for a matrix to be in reduced row echelon form (RREF).

Solution. \( m \times n \) matrix \( R \) is in RREF means

(a) \( R \) is in echelon form.

(b) Every pivot is 1.

(c) Columns with a pivot have no other nonzero entry.

2. TRUE or FALSE? (No need for explanation):

(a) Every upper-triangular matrix is in reduced row echelon form?

(b) Every lower-triangular matrix is in reduced row echelon form?

(c) Every permutation matrix is in reduced row echelon form?

(d) The following matrix is in reduced row echelon form?

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 2 \\
\end{bmatrix}
\]

(e) The reduced row echelon form of \( A \) is unique?

(f) The full solution set of \( Ax = b \), where \( A \) is \( m \times n \) and \( b \in \mathbb{R}^m \), is always a vector subspace of \( \mathbb{R}^n \)?

(g) The difference \( a = x_1 - x_2 \), between any two solutions \( x_1 \) and \( x_2 \) to \( Ax = b \), is a vector that belongs to the null space \( N(A) \)? (Apply the rule \( A(x + \lambda y) = Ax + \lambda Ay \) to \( A(x_1 - x_2) \) to answer the question).

Solution. (a) No. The rows of all zeros must be below all the other rows. This is not true, for instance, of

\[
\begin{bmatrix}
0 & 0 \\
0 & 1 \\
\end{bmatrix}
\]

(b) No. For instance,

\[
\begin{bmatrix}
0 & 0 \\
1 & 0 \\
\end{bmatrix}
\]

is not.
(c) No. For instance, for the matrix
\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]
the pivot of the second row is to the left of the pivot of the first row.

(d) No. The leading coefficient of the second row is not a one.

(e) Yes. This will be explained in class, though you do not need to know a proof. (The proof-oriented reader should read e.g. http://web.gccaz.edu/wkeowsk/225-Linear-10-11-Sp/yuster-rref-unique.pdf.)

(f) No. For instance, the solution set of
\[
\begin{bmatrix}
1 & 2 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
2 \\
0
\end{bmatrix}
\]
contains a unique vector, \([0, 1]^T\). This is not a vector subspace of \(\mathbb{R}^2\).

(g) Yes. \(A(x_1 - x_2) = A(x_1) - A(x_2) = b - b = 0\).

\[\Box\]

3. Do Problems 20 & 23 from Section 3.2.

**Solution to 3.2.20:**

Let \(A\) be the matrix in the problem.

The column 5 does not have pivot. If not, since \((A)_{45} = c \neq 0\) is a pivot and \((A)_{4i} = 0\) for any \(i \neq 5\), column 1 + column 3 + column 5 = \((*,*,*,*)^T \neq 0\). In other words, the fifth variable \(x_5\) is the only free variable. We have

\[
A \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \text{column 1} \cdot 1 + \text{column 3} \cdot 1 + \text{column 5} \cdot 1 = 0.
\]

Hence the special solution is \((1, 0, 1, 0, 1)^T\) and the null space is \(\{(x_5, 0, x_5, 0, x_5)^T : x_5 \in \mathbb{R}\}\).

**Solution to 3.2.23:**

\((a, b, c) = \left( -\frac{1}{2}, -2, -3 \right)\)

satisfies the equation
\[
\begin{bmatrix}
1 & 0 & a \\
1 & 3 & b \\
5 & 1 & c
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
2
\end{bmatrix} = 0.
\]
Hence
\[
\begin{bmatrix}
1 & 0 & -\frac{1}{2} \\
1 & 3 & -2 \\
5 & 1 & -3
\end{bmatrix}
\]
is a matrix we wanted.

4. Do Problem 35 from Section 3.2.

*Solution.* The nullspace of \( B = [AA] \) contains all vectors \( x = \begin{bmatrix} y \\ -y \end{bmatrix} \) for all \( y \) in \( \mathbb{R}^4 \).

5. Do Problems 3 & 8 from Section 3.3.

*Solution to 3.3.3:*

\[
A = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 3 \\
2 & 4 & 6
\end{bmatrix} \rightarrow \begin{bmatrix}
2 & 4 & 6 \\
0 & 0 & 3 \\
0 & 0 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 2 & 3 \\
0 & 0 & 3 \\
0 & 0 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 2 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix} = \text{RREF}(A).
\]

Using the same elimination or permutation operators as in the case \( A \), we get

\[
\text{RREF}(B) = [\text{RREF}(A)\text{RREF}(A)].
\]

\[
C = \begin{bmatrix}
A & A \\
A & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
A & A \\
0 & -A
\end{bmatrix} \rightarrow \begin{bmatrix}
A & 0 \\
0 & -A
\end{bmatrix} \rightarrow \begin{bmatrix}
A & 0 \\
0 & A
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{RREF}(A) & 0 \\
0 & \text{RREF}(A)
\end{bmatrix} = \text{RREF}(C).
\]

*Solution to 3.3.8:*

If the matrix has rank 1, every column is constant multiple of any other nonzero columns. So

\[
A = \begin{bmatrix}
1 & 2 & 4 \\
2 & 4 & 8 \\
4 & 8 & 16
\end{bmatrix}, B = \begin{bmatrix}
3 & 9 & -9 \\
1 & 3 & -3 \\
2 & 6 & -3
\end{bmatrix}.
\]

For \( M \), if \( a \neq 0 \),

\[
M = \begin{bmatrix}
a & b \\
c & \frac{b+c}{a}
\end{bmatrix}
\]

and if \( a = 0 \),

\[
M = \begin{bmatrix}
0 & b \\
0 & d
\end{bmatrix}
\]

for any \((b,d) \neq (0,0)\), or \( M = \begin{bmatrix}
0 & 0 \\
c & d
\end{bmatrix} \) for any \((c,d) \neq (0,0)\).
6. Do Problems 17 & 28 from Section 3.3.

Solution to 3.3.17:

(a) By matrix multiplication, each column of $AB$ is $A$ times the corresponding column of $B$. So if column $j$ of $B$ is a combination of earlier columns, then column $j$ of $AB$ is the same combination of earlier columns of $AB$. Thus rank $(AB) \leq$ rank $(B)$. There are no new pivot columns!

(b) The rank of $B$ is $r = 1$. Multiplying by $A$ cannot increase this rank. The rank of $AB$ stays the same for $A_1 = I$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. It drops to zero for $A_2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$.

Solution to 3.3.28:

The row-column echelon form is always $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$; $I$ is the $r \times r$ identity matrix.

7. Do Problems 5 & 16 from Section 3.4.

Solution to 3.4.5: Consider the augmented matrix

$$
\begin{bmatrix}
1 & 2 & -2 & b_1 \\
2 & 5 & -4 & b_2 \\
4 & 9 & -8 & b_3
\end{bmatrix}
$$

The operations those make the first $3 \times 3$ matrix to RREF change our augmented matrix to

$$
\begin{bmatrix}
1 & 0 & -2 & 5b_1 - 2b_2 \\
0 & 1 & 0 & -2b_1 + b_2 \\
0 & 0 & 1 & -2b_1 - b_2 + b_3
\end{bmatrix}
$$

Hence this equation is solvable when $-2b_1 - b_2 + b_3 = 0$ and the set of solutions is \{(5b_1 - 2b_2 + 2z, -2b_1 + b_2 + z) : z \in \mathbb{R}\}.

Solution to 3.4.16:

The largest possible rank of a $3 \times 5$ matrix is 3. Then there is a pivot in every row of $U$ and $R$. The solution of $Ax = b$ always exists. The column space of $A$ is $\mathbb{R}^3$. An example of $A$ is

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}
$$

8. Do Problems 24 & 33 from Section 3.4.

Solution to 3.4.24:

(a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix} [x] = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ has 0 or 1 solutions, depending on $b$. 

4
(b) \[
\begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= [b]
\] has infinitely many solutions for every \( b \).

(c) There are 0 or \( \infty \) solutions when \( A \) has rank \( r < m \) and \( r < n \): the simplest examples is a zero matrix.

(d) One solution for all \( b \) when \( A \) is square and invertible (like \( A = I \)).

Solution to 3.4.33:
If the complete solution to \( A\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \) is \( \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c \end{bmatrix} \) then \( A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \).

9. Do Problems 9 from Section 3.5.

Solution. (a) the dimension of \( \mathbb{R}^3 \) is 3 and 3 is the biggest possible number of independent vectors in \( \mathbb{R}^3 \).

(b) there exists \((c_1, c_2) \neq (0, 0)\) such that \( c_1 \cdot v_1 + c_2 \cdot v_2 = 0 \).

(c) \( 0 \cdot v_1 + 1 \cdot (0, 0, 0) = (0, 0, 0) \).

(See Problem 10 on next page!)
10. In this exercise, we try MATLAB’s function \texttt{null(A)} for finding a basis (i.e. a minimal set of spanning vectors = a maximal set of independent vectors) for the null space of a matrix. We also try \texttt{rref(A)} for finding the reduced row echelon form.

\begin{verbatim}
B = [1 0 0 0; 
    0 0 1 0; 
    0 0 0 1; 
    0 1 0 0];

C = [1 2 1 -2; 
    0 0 1 5; 
    0 0 0 0; 
    0 0 0 0];

D = [1 2 0 1; 
    0 2 2 1; 
    0 0 3 3; 
    1 0 0 4];
\end{verbatim}

(a) Using \texttt{null()}, find a basis of each of $N(B), N(C)$ and $N(D)$ (the column vectors in the matrix MATLAB outputs are the basis vectors). Same for $N(BC)$ and $N(DC)$.

(b) Figure out whether $N(C)$ and $N(DC)$ are the same subspaces of $\mathbb{R}^4$, as follows:

MATLAB can easily perform this, if we make use of the following two facts, for $V$ and $W$ subspaces of $\mathbb{R}^n$ with given collections of vectors used for spanning them, respectively $v_1, \ldots, v_k$ spanning $V$ and $w_1, \ldots, w_l$ spanning $W$.

\textbf{Fact 1:} A vector $b \in \mathbb{R}^4$ belongs to $V$ if and only if the system $Ax = b$ has at least one solution, where $A = [v_1 \ v_2 \ldots \ v_k]$ is the matrix which as columns has a collection of vectors we use to span $V$.

\textit{Example} ($2 \times 2$): In MATLAB we create the augmented matrix $[A|b]$ and use the command \texttt{rref}.

\begin{verbatim}
A = [1 2; 
    -1 -2];

b = [1; 
     1];

>> A_aug_b = [A b]

A_aug_b =
    1  2  1
    -1 -2 1
\end{verbatim}
>> rref(A_aug_b)

ans =
     1     2     0
     0     0     1

(Note: A_aug_b is only a variable name. The augmentation bars in the output will not show in MATLAB).

Notice the zero row that has a non-zero entry to the right of the bar: This system $Ax = b$ has no solution. Hence, $b = [1, 1]^T$ is not in the subspace spanned by the columns of $A$.

**Fact 2:** Two subspaces are the same, $V = W$, if and only if:

i. Vectors spanning $V$ lie in $W$, that is $v_1, \ldots, v_k \in W$ (so $V \subseteq W$), and

ii. Vectors spanning $W$ lie in $V$, that is $w_1, \ldots, w_k \in V$ (so $W \subseteq V$).

**Example:** Referring to the previous example, the subspace $V$ spanned by the vectors $b$ and $[0, 1]^T$ cannot be the same as the subspace $W$ spanned by the columns of $A$ (since we saw $b \notin W$).

Now, for using Fact 1 & Fact 2 in MATLAB to determine if $N(C)$ and $N(DC)$ are in fact the same, you will need the ":=" option:

>> A(:,2) %Example: Gives you the 2nd column from matrix A

Then proceed as in the examples, checking each basis vector from one space for membership of the other space.

(c) Which property of the square matrix $D$ explains the result of your comparison of $N(C)$ and $N(DC)$? State this as a general rule, and put a box around it. Apply your rule to explain why $N(DC)$ and $N(BC)$ are the same subspace.

(d) Is $N(CB)$ the same as $N(C)$? Either use the method from (b) again (you can do it all at once using rref([null(CB) null(C)]), if you carefully read off the result!), or simply try applying $CB$ to the basis vectors you found for $N(C)$, and vice versa.

**Solution.**

(a) Bases for the null spaces are as follows:

>> null(B)

ans =

Empty matrix: 4-by-0

>> null(C)

ans =

     0    0.9245
     0.5659    -0.3142
>> null(D)

ans =

Empty matrix: 4-by-0

>> null(B*C)

ans =

0    -0.9245
0.5659    0.3142
-0.8085    0.2115
0.1617   -0.0423

>> null(D*C)

ans =

-0.0331    0.9239
-0.5543   -0.3343
0.8155   -0.1824
-0.1631    0.0365

(b) Here we get, for example:

nC=null(C);
nC_1 = nC(:,1);
nC_2 = nC(:,2);

nDC = null(D*C);

redux1 = rref([nDC nC_1])
redux2 = rref([nDC nC_2])

>> redux1 =

1.0000    0    |   -0.9994
0    1.0000    |   -0.0358
0    0    |   0
0    0    |   0
Here we looked at the system $Ax = b$, with $A$’s columns being the basis of $N(DC)$ and for $i = 1, 2$ let $b_1 = nC_1$, $b_2 = nC_1$ be the two basis vectors we got for $N(C)$ in (a). Since in both cases the system is consistent (the zero rows in the left compartment of the above RREF of the augmented matrix has a corresponding zero in the right compartment).

Thus, since $b_1, b_2 \in N(DC)$ and since $N(C)$ is spanned by its two basis vectors $b_1, b_2$, we conclude that: $N(C) \subseteq N(DC)$.

Note: $N(C) \subseteq N(DC)$ is true for any matrices $D$ and $C$! Why? Because if $x \in N(C)$, meaning $Cx = 0$, then also $DCx = DO = 0$ meaning $x \in N(DC)$.

Similar code, reversing the roles of $C$ and $DC$ checks for us that also (which is not always true - see below): $N(DC) \subseteq N(C)$.

Thus, we have checked that: $N(DC) = N(C)$.

(c) The property the square matrix $D$ has is: Invertible. Here’s the rule:

\[
\text{If } D, C \text{ are any } n \times n \text{ matrices, and } D \text{ invertible, then } N(DC) = N(C).
\]

We saw the invertibility of $D$ above in (a): The basis for the null space was $\emptyset$ (the empty set), so $N(D) = \{0\}$ (the subspace only consisting of the zero vector). Thus, if we reduced $D$ to its RREF matrix $R$ we would obtain the $4 \times 4$ identity $I$ (since $D$ is square!). But this means that $D$ is invertible.

This also explains why $B$ is invertible, using (a). Now, we may use our new rule: $N(DC) = N(DB^{-1}B)C = N(DB^{-1}BC) = N(BC)$.

(d) No, $N(CB)$ and $N(C)$ are not the same (in this example).

```
nC=null(C);
nCB=null(C*B);
BigMat = rref([nCB nC]);
```

```
BigMat =

1 0 | 0 0
0 1 | 0 0
0 0 | 1 0
0 0 | 0 1
```

Note that we have solved all the four systems at once by using the augmentation. Reading left-to-right, you can see that none of the two basis vectors MATLAB chose for us for $N(C)$ belong to $N(CB)$. Reading right-to-left, we see that
reversely the $N(CB)$ basis we chose is not in $N(C)$. So these subspaces are not identical.
Alternatively, we can try:

```matlab
C*B*nC_1
```

```
ans =

-2.5870
2.9913
0
0
```

Since that’s not the zero vector, we have that the vector $nC_1$ from $N(C)$ is not in $N(CB)$.
So, these two subspaces are not the same. □