18.06 Spring 2012 – Problem Set 6

This problem set is due Thursday, April 5th, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

Every problem is worth 10 points.

1. Do Problems 9 & 15 from Section 5.1.

   **Solution. Problem 9**
   
   det(A) = 1: exchange row 1 and row 3, and then row 1 and row 2.
   
   det(B) = 2: subtract rows 1 and 2 from row 3 then columns 1 and 2 from column 3.
   
   det(C) = 0: the rows are equal. (Note that C = A + B, but det(C) ≠ det(A) + det(B).)

   **Problem 15**
   
   The first determinant is zero: subtract row 2 from row 3, and row 1 from row 2, to get a matrix with two equal rows.
   
   The second determinant is $(1 - t^2)^2 = 1 - 2t^2 + t^4$: subtract $t$ times row 2 from row 1, and $t$ times row 3 from row 2, to get a lower-triangular matrix.

2. Do Problems 18 & 22 from Section 5.1.

   **Solution. Problem 18**
   
   
   
   
   
   where to reach the $2 \times 2$ determinant, we eliminate $a$ and $a^2$ in row 1 by column operations. Now factor out $b - a$ and $c - a$ from the $2 \times 2$ determinant:

   $$(b - a)(c - a) \begin{vmatrix} 1 & b + a \\ 1 & c + a \end{vmatrix} = (b - a)(c - a)(c - b).$$

   **Problem 22**
   
   det(A) = 3, det($A^{-1}$) = 1/3, det($A - \lambda I$) = $\lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$. The numbers $\lambda = 1$ and $\lambda = 3$ give det($A - \lambda I$) = 0.

3. Do Problems 8 & 9 from Section 5.2.
Solution. Problem 8

Some term $a_{i_1}a_{i_2} \ldots a_{i_n}$ in the big formula is not zero. Move rows 1, 2, \ldots, $n$ into rows $i_1$, $i_2$, \ldots, $i_n$. Then these non-zero $a$'s will be on the main diagonal.

Problem 9

There are 6 terms in the big formula, all $\pm 1$. Thus, the determinant must be an even integer.

To get +1 for all the three product terms corresponding to the even permutations, the matrix needs to have an even total number of $-1$ entries (this is easy to see in this $3 \times 3$ situation, where $3 \cdot 3!/2 = 3^2$ happens to hold, so each matrix entry shows up exactly once somewhere in the 3 product terms coming from the even permutations). On the other hand, to also get +1 for all the product terms corresponding to the odd permutations, the matrix would need to have an odd total number of $-1$ entries.

Thus at least one term in the big formula must be a $-1$, and the maximal determinant is $+4$. Namely, this is attained for example for the matrix:

$$
\begin{bmatrix}
1 & 1 & 1 \\
1 & -1 & 1 \\
1 & 1 & -1
\end{bmatrix}.
$$

4. Do Problem 20 from Section 5.2.

Solution. $G_2 = -1$, $G_3 = 2$, and $G_4 = -3$. We guess that $G_n = (-1)^{n-1}(n-1)$.

5. Do Problem 29 from Section 5.2.

Solution. There are five non-zero products, all $\pm 1$. Here are the (row, column) coordinates of the terms, and the signs:

$$
+ \quad (1,1)(2,2)(3,3)(4,4) \\
+ \quad (1,2)(2,1)(3,4)(4,3) \\
- \quad (1,2)(2,1)(3,3)(4,4) \\
- \quad (1,1)(2,2)(3,4)(4,3) \\
- \quad (1,1)(2,3)(3,2)(4,4)
$$

Total: $-1$.

6. Do Problem 34 from Section 5.2.

Solution. (a) Consider the 3 by 5 matrix formed by the last three rows of $A$. It has only two non-zero columns, and so it has rank at most 2. Therefore, the last three rows of $A$ are linearly dependent.
(b) Consider a term in the big formula for \( \det A \); it is a product of entries of \( A \), one entry in each row and column. Consider the entries coming from the last three rows of \( A \); there are three of them, and at most two can be in the last two columns of \( A \). Therefore, at least one entry falls in the 3 by 3 block of zeroes, and so the whole term of the big formula is 0.

7. Do Problems 4 & 8 from Section 5.3.

\textit{Solution. Problem 4}

(a) We get the familiar formula \( x_1 = |b \ a_2 \ a_3|/\det(A) \)

(b) We use linearity of the determinant in the first column:

\[ |x_1a_1 + x_2a_2 + x_3a_3 a_2 \ a_3| = x_1|a_1 \ a_2 \ a_3| + x_2|a_2 \ a_2 \ a_3| + x_3|a_3 \ a_2 \ a_3| \]

The last two terms are zero because two of the columns are the same.

\textbf{Problem 8} The cofactor matrix is

\[
C = \begin{bmatrix}
6 & -3 & 0 \\
3 & 1 & -1 \\
-6 & 2 & 1
\end{bmatrix}
\]

Then \( AC^T = 3I \) so \( \det(A) = 3 \). If you change that 4 to 100, \( \det(A) \) is unchanged because the corresponding cofactor is 0.

8. Do Problems 20 & 25 from Section 5.3.

\textit{Solution. Problem 20} The rows of the Hadamard matrix generate a 4-dimensional hypercube with side length 2. Therefore \( \det(H) = \pm 16 \). It turns out that the rows of \( H \) are in such an order that in fact \( \det(H) = 16 \).

\textbf{Problem 25} An \( n \) dimensional cube has \( 2^n \) corners, \( n \ 2^{n-1} \) edges and \( 2n \ (n-1) \)-dimensional faces. The cube in \( \mathbb{R}^n \) generated by the rows of \( 2I \) has volume \( 2^n \).

9. Do Problem 1 from Section 6.1.

\textit{Solution.} The eigenvalues of \( A \) are 1 and 1/2; the eigenvalues of \( A^2 \) are 1 and 1/4; the eigenvalues of \( A^\infty \) are 1 and 0.

(a) If we swap the rows of \( A \), the resulting matrix has eigenvalues 1 and -1/2

(b) A matrix has a zero eigenvalue if and only if it has a non-trivial nullspace. The nullspace of a matrix is not changed by the steps of elimination; therefore, a zero eigenvalue is not changed by the steps of elimination.
10. Use MATLAB to "prove" all the facts you remember (or may not remember?) about determinants. First define the following matrices to test on (copy paste into MATLAB - retyping is time- and patience-consuming):

```
%Two random 4 x 4 matrices:
A = rand(4,4);
B = rand(4,4);

%An elementary subtraction of rows (by left-multiplying. Of columns if you right-)
E = [1 -3 0 0;
    0 1 0 0;
    0 0 1 0;
    0 0 0 1];

%An "odd" permutation:
P_odd = [1 0 0 0;
        0 1 0 0;
        0 0 0 1;
        0 0 1 0];

%An "even" permutation:
P_even = [0 1 0 0;
         1 0 0 0;
         0 0 0 1;
         0 0 1 0];

%Another 4 x 4...almost, the 1st row is missing:
C = rand(3,4);

%Two random row vectors
a1 = rand(1,4);
a2 = rand(1,4);

%Two matrices having the a_1, a_2 as 1st rows
D1 = [a1;
      C ]
D2 = [a2;
      C ]

%Matrix with sum as 1st row
D = [a1 + a2;
     C ]
```
Now, using these matrices do the following tests. We’ve slipped in a couple of *false* ones - to make it more exciting (take a guess before you hit `<enter>`).

(a) \( \det(D_1) + \det(D_2) = \det(D) \)
(b) \( \det(A) + \det(B) = \det(A + B) \)
(c) \( \det(10 \times A) = 10 \times \det(A) \)
(d) \( \det(E \times A) = \det(A) = \det(A \times E) \)
(e) \( \det(P_{\text{odd}} \times A) = -\det(A) \)
\(\det(P_{\text{even}} \times A) = \det(A) \)

Which ones in (a)-(e) are correct, and which are false?

*Solution.* (a) True
(b) False
(c) False
(d) True
(e) True

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**18.06 Wisdom.** Enjoyed your spring break? True!